<u>Mathematics for Life Sciences</u>

<u>An Overview</u>

Sometimes in the quick pace of a math class it is easy to lose track of why we are learning the materials. I hope the following will illustrate some of the reasons why math 151-152 was created for life science students. I also want to not only provide some insight into *<u>what</u>* we will learn this semester in math 152, but *<u>why</u>* we are learning it.

Life sciences students and/or health professionals concern themselves with the study of a (or various) <u>Biological Process(es)</u>

Some Examples:

There are two ways to study any given biological process

a. Qualitatively- Making observations/descriptions using words
b. Quantitatively- Making observations using numbers. Collecting data

We are approaching life sciences from the quantitative point of view and therefore we have to consider data.

Making sense of data is where <u>statistics and probability</u> comes in

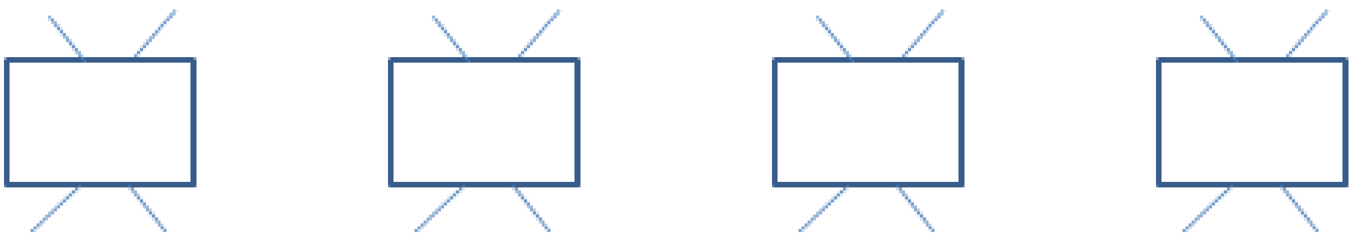
- Analyzing data (mean, mode, median, distribution etc..)
- Fitting data (least squares etc..)
- What are the chances an event will occur?
    - Does history matter?
- Genetics- what traits will offspring have?
- Population dynamics
    - Birth/death rate
    - Disease spread

Once we have an idea of how a biological process works we can use mathematics to model it.

Mathematical model: A description of a system using mathematical concepts and language. We will focus on mathematical models that are functions.

Function: A rule that assigns to each element x in one set (known as the Domain) exactly one element, $f(x)$, in another set (known at the Range)

Domain can be thought of as a set of input values where the range is the set of possible output values



The graph of a function is the set of all ordered pairs $(x, f(x))$ where x is in the domain.

Main different between functions in 151 compared to those in 152:

Consider the following hypothetical continuous mathematical models:

1) Let $Q(t)$ represent the quantity of a drug in the blood stream t minutes after ingesting it
2) Let $A(s)$ represent the amount of algae in a pond given sunlight level s
3) Let $W(r)$ represent the number of Wolves in the forest as a function of the number of rabbits, r.

Possible topics of interest for above models:

- Knowing what happens at a point of interest

- What happens in the long run?

- Both previous examples can be posed as a rate of change question

- Knowing total amounts

Chapter 15- Limits

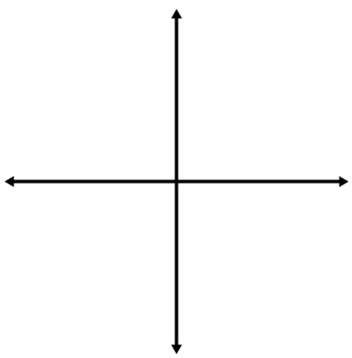
$\lim_{x \to a} f(x) = L$

- reads: the limit of $f(x)$ as x approaches a is L
- exists if $f(x)$ gets arbitrarily close to L as x gets arbitrarily close to a

Theorem: $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$

3 Ways to Evaluate a Limit:

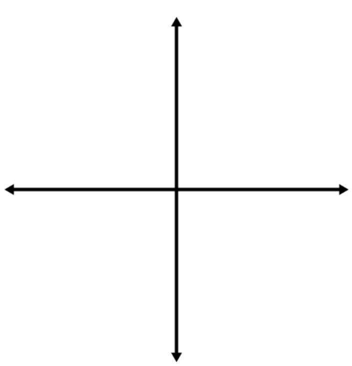
1) 2) 3)



Sometimes you can just plug $x = a$ into $f(x)$
However, not always:

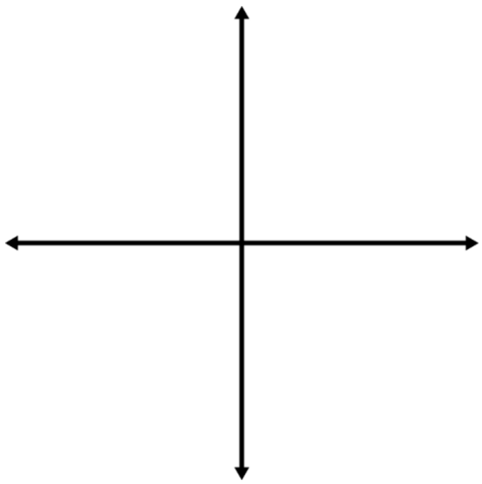
EX: Let $(t) = \frac{\sqrt{t^2+9}-3}{t^2}$. Consider $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

1) 2) 3)



Some limits do not exist:

<u>EX:</u> $\lim_{x \to 0} \frac{|x|}{x}$



<u>Evaluating Limits Algebraically</u>

<u>Key Properties of Limits:</u>

Let $a,\ c,\ n,\ L\ and\ M$ be real numbers
Let $\lim_{x \to a} f(x) = L$ and let $\lim_{x \to a} g(x) = M$

1) $\lim_{x \to a}\ c = c$
2) $\lim_{x \to a}\ f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$
3) $\lim_{x \to a}\ f(x) - g(x) = L - M$
4) $\lim_{x \to a}\ f(x) \cdot g(x) = L \cdot M$
5) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$
6) $\lim_{x \to \pm\infty}\ x^{-n} = \lim_{x \to \infty} \frac{1}{x^n} = 0$ for $n > 0$
7) $\lim_{x \to \pm\infty}\ e^{-nx} = \lim_{x \to \infty} \frac{1}{e^{nx}} = 0$ for $n > 0$

Use the above properties whenever possible to evaluate limits of sums, differences, products and quotients.

However, they don't always work right away

<u>EX:</u> $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

Indeterminate Forms

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \infty - \infty \qquad \infty \cdot 0 \qquad 0^0 \qquad \infty^0 \qquad 1^\infty$$

If you using one of the key properties on the previous page results in an indeterminate form you need to rethink your approach.

Tools for dealing with indeterminate forms:

a) Multiply by an appropriate form of 1

   a. Usually either $\frac{\text{Conjugate}}{\text{Conjugate}}$ or $\frac{\frac{1}{x^n}}{\frac{1}{x^n}}$ for an appropriate value of n

b) Factor and cancel common terms

c) Combine terms

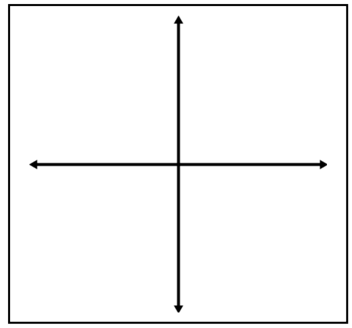
EX: Lets revisit $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

EX: $\lim_{x\to 1} \frac{1}{x-1} - \frac{2}{x^2-1}$

Limits at Infinity

- All key limit properties still apply
- Fact: If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ then we say that $f(x)$ has a horizontal asymptote at $y = L$

EX: From "key properties of limits", $\lim_{x\to\pm\infty} \frac{1}{x} = 0$



EX:

From chapter 5, the photosynthetic rate for the lower leaves on a soybean plant is a function of the light level:

$$P(I) = \frac{1.62I}{0.195I + 8.29} - 3.13$$

Where P is the photosynthetic rate and I is the light level measured in $\mu\text{mol m}^{-2}\, s^{-1}$.

Estimate the maximum photosynthetic rate for lower leaves on a soybean plant.

Ex: $\lim_{x \to \infty} \frac{3x}{\sqrt{x^2+8}}$

Ex: $\lim_{x \to 2} \frac{x^2-4}{x-2}$

Ex: $\lim_{x \to 16} \frac{\sqrt{x}-4}{x-16}$

Chapter 16- Continuity

Let $f(x)$ be a function, let "a" and "L" be real numbers.

In chapter 15 we saw the idea of left and right side limits, ie, $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.

Theorem

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$

Continuity

Let $f(x)$ be a function with "a" in its domain.

We say $f(x)$ is continuous at a if $\lim_{x \to a} f(x) = f(a)$

This implies: 1)

2)

3)

4)

- If a function is continuous at every point in its domain, then entire function is said to be continuous
- If the function is not continuous at a point "a", we call it discontinuous at $x = a$

SOME EXAMPLES:

Continuous functions

Discontinuous Functions

An Infinite Discontinuity occurs when $\lim_{x \to a^{\pm}} f(x) = \pm\infty$

EX: $\ln(x)$ and $\frac{1}{x}$ at $x = 0$

A Jump Discontinuity occurs when $\lim_{x \to a^{+}} f(x)$ and $\lim_{x \to a^{-}} f(x)$ both exist but are not equal

EX: $\frac{|x|}{x}$

A Removable Discontinuity occurs at a point in the graph where a hole exists

EX: $\frac{x^2-1}{x-1}$

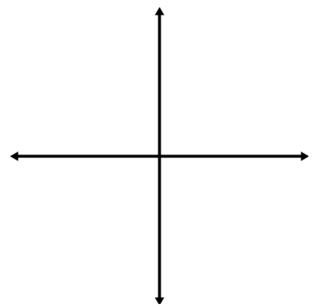
Two types of functions we will encounter in math 152 <u>may</u> not be continuous:

a) Rational functions

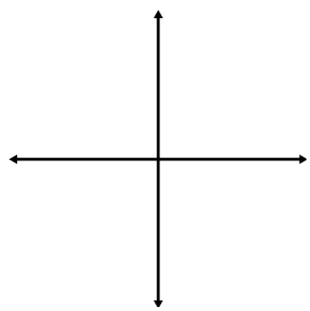
b) Piecewise functions

<u>Rational Functions with Discontinuities:</u>

<u>EX:</u> $f(x) = \frac{1}{x-1}$

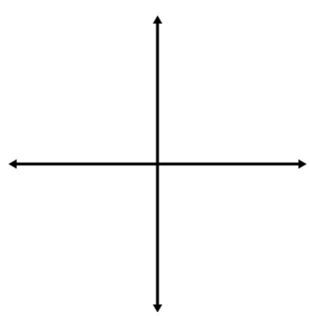


<u>EX:</u> $g(x) = \frac{x^2-1}{x+1}$



<u>Piecewise Function with a Discontinuity:</u>

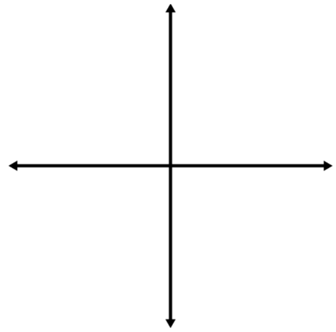
<u>EX:</u> $h(x) = \begin{cases} 1 & x < 0 \\ x & x > 0 \end{cases}$



<u>Note:</u> Not all functions of these forms have obvious discontinuities → see next page

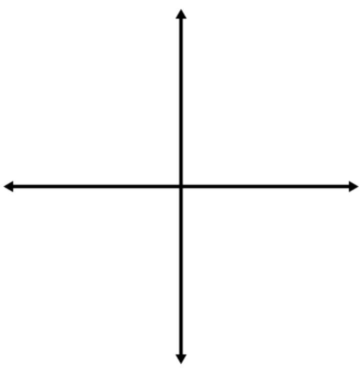
Rational functions always have discontinuities at the values that make their denominator equal zero. However, the specific type of discontinuity is not always obvious:

EX: $f(x) = \frac{\tan(x)}{x}$



EX of a continuous piecewise function:

$$h(x) = \begin{cases} 1 & x \leq 0 \\ x + 1 & x > 0 \end{cases}$$



EX

Intermediate Value Theorem

Often times we are interested if a function achieves a certain value.

For example, suppose a population of insects can be modeled by

$$P(t) = \frac{t^2}{2} - \frac{t^3}{1500} + 6$$

Where t is measured in days and P has units number of individuals.

A valid and meaningful question would be whether or not the population achieves a certain value.
For example, we might be interested if there is ever a time that the population ever reaches 5000 individuals. That is, does there exist a t^* such that $P(t^*) = 5000$?

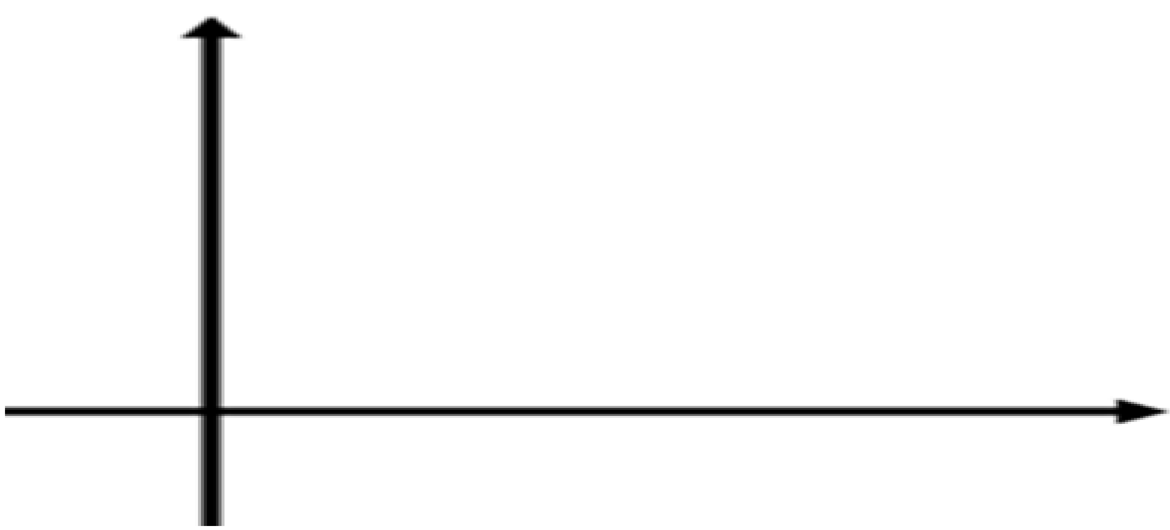
Option #1: Solve $\frac{t^2}{2} - \frac{t^3}{1500} + 6 = 5000$ for t

Option #2: Graph $P(t)$

Option #3: If we don't care about the exact value of t^* that yields $P(t^*) = 5000$ but instead just want to know if the population ever reaches 5000 then we can use the intermediate value theorem:

The Intermediate Value Theorem

If $f(x)$ is a function that is continuous on a closed interval $[a, b]$ and N is some value between $f(a)$ and $f(b)$, then there exists and value c such that $a < c < b$ and $f(c) = N$



EX: Let $P(t) = \frac{t^2}{2} - \frac{t^3}{1500} + 6$ model a population of insects as described on the previous page. Use the Intermediate Value Theorem to prove that the population achieves a size of 5000.

EX: The photosynthetic rate of a 215 day old soybean plant can be modelled by

$$P(t) = -0.237t^2 + 5.01t - 6.58$$

Where P is the photosynthetic rate measured in $\mu\text{mol m}^{-2}\, s^{-1}$ and t is measured in hours.

Use the Intermediate Value Theorem to prove that there exists a time of day such that a 215 day old soybean plant has a photosynthetic rate of 10.

Chapter 17- Rates of Change

When considering a biological system, a researcher often wants to know where the system *will be* rather than *where it currently is.* Thus, to understand a biological situation, it is often important to understand how the given system is changing rather than trying to consider its current state.

Some examples:

- How does the weight of a moose change…
  a) Over the first 10 years of its life
  b) When the moose is 200 hours old

- How is the concentration of a drug in a person's blood stream changing…
  a) Over the course of the first several hours after ingesting it
  b) 25 minutes after ingesting the drug.

- What is the rate of change of federal spending on stem cell research…
  a) Over the past 10 years
  b) In 2007

Different Types of Rates of Changes

Type a) above are all____. These types of rates describe how something has changed over a period rather than at a specific value. This type of rate of change gives more general information about the system in question.

Ex:

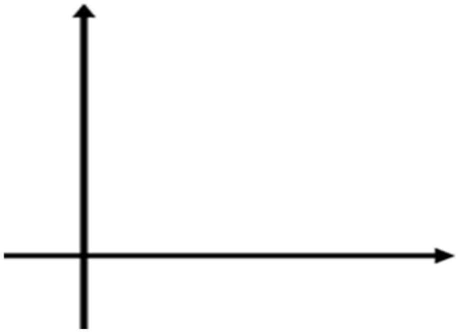
Type b) above are all____, because they describe what is happening at a specific point (or instance). This type gives specific information at a point rather than general information over a period. The instantaneous rate of change of a function $f(x)$ at a point "a" is called *the derivative of* $f(x)$ *at a*, denoted $f'(a)$.

Ex:

A) Average Rate of Change- Tells us general information about how something has changed over a period

The average rate of change of quantity $f(x)$ over the interval $[x_1, x_2]$ is given by the following:

$$\frac{\text{change in } f(x)}{\text{change from } x_1 \text{to } x_2} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



EX: The weight of a female Moose at time t is determined by $w(t) = 369(0.93)^t(t^{0.36})$, where w is weight (in kg) and t is time (years). What is the average rate of change of the weight of a female Moose from age 1 to age 3?

Average velocity is a common application of average rate of change

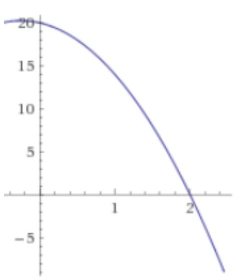
Using the above definition of average rate of change, Average velocity $= \frac{\text{change in position}}{\text{change in time}}$

EX: Suppose we drove to Nashville (~180 miles) in 3 hours. Find the average velocity of the trip.

EX: A King Fisher's height h (in feet) from the water at time t (in seconds) is modelled by:

$$h(t) = -4t^2 - 2t + 20$$

Find the bird's average velocity over the first 2 seconds.



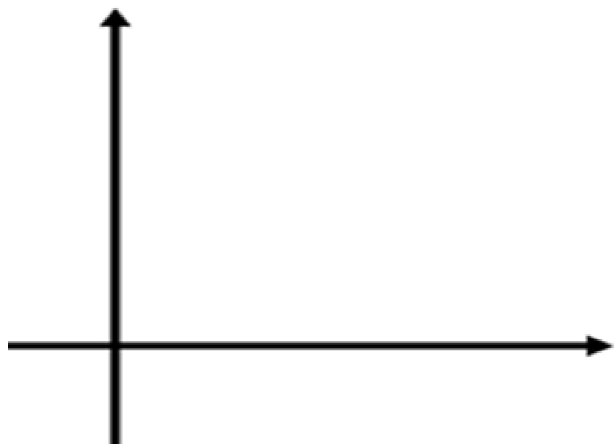
B) Instantaneous Rate of Change- tells us the rate of change at a specific point. If we are given a function this can be thought of as the slope (or steepness) of the function at a specific point.

There are 2 methods for finding instantaneous rate of change. One is an estimate, one is exact.

In all of the following, assume that $a < b < c$

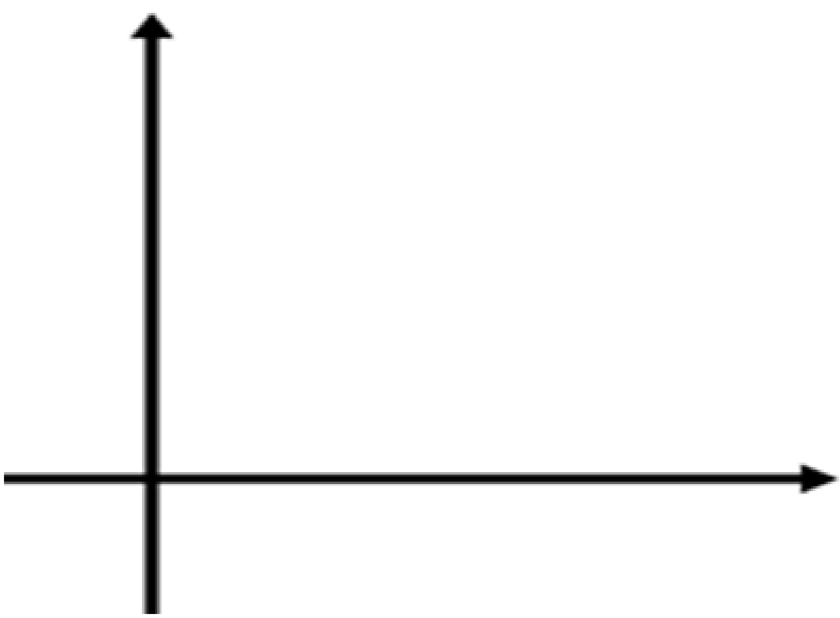
Method 1 (an estimate):

$$f'(b) \approx \frac{\frac{f(b)-f(a)}{b-a} + \frac{f(c)-f(b)}{c-b}}{2} = \frac{1}{2}\left(\frac{f(b)-f(a)}{b-a} + \frac{f(c)-f(b)}{c-b}\right)$$



Method 2 (the exact value):

$$f'(b) = \lim_{x \to b} \frac{f(x)-f(b)}{x-b}$$



EX: Suppose $c(t)$ models the concentration of a drug in the blood stream (in ppm) t minutes after injection. The following table gives values of c at various times t:

t	0	0.1	0.2	0.3	…
$c(t)$	0.84	0.89	0.94	0.98	…

Estimate the instantaneous rate of change of the drug at $t = 0.2$ minutes.

EX: A King Fisher's height h (in feet) from the water at time t (in seconds) is modelled by:

$$h(t) = -4t^2 - 2t + 20$$

a) Estimate the velocity at $t = 1$
b) Exactly calculate the velocity at $t = 1$

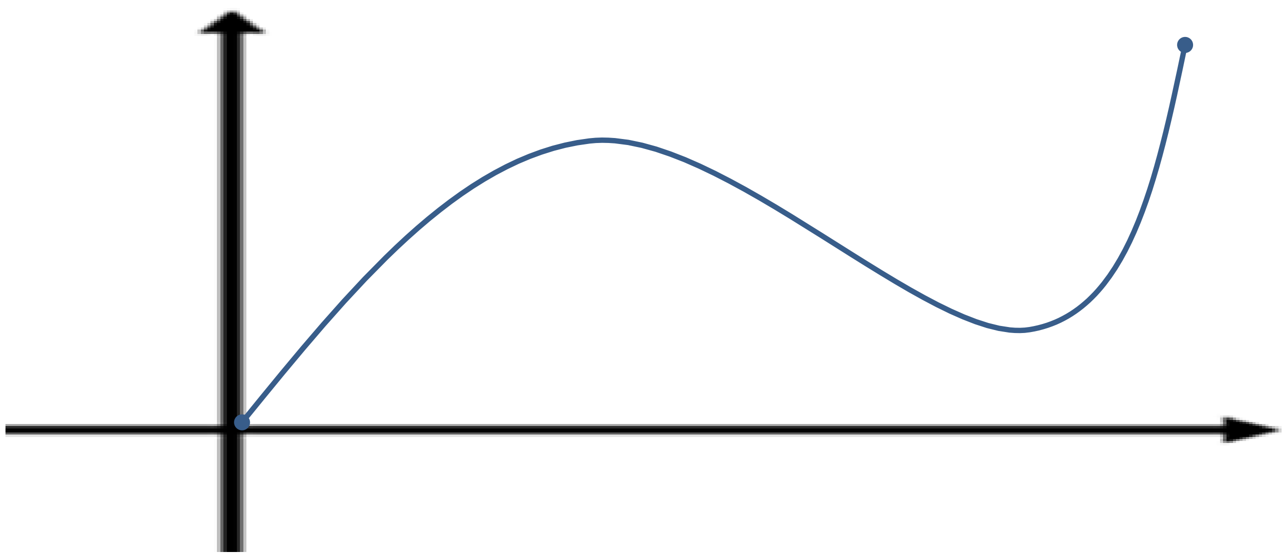
Hint/reminder: velocity is a rate of change

Visualizing Rate of Change

Since rates of change tell us how a function is changing…

- A positive rate of change at $f(a)$ means the graph is increasing at that point
- A negative rate of change at $f(a)$ means the graph is decreasing at that point
- Zero rate of change occurs where the graph is "flat"

EX: Suppose we drive from Knoxville to Nashville and that the following graph depicts the cars distance from Knoxville during our trip.



- Where is the rate of change the largest?
- Where is there zero rate of change?
- What is happening at 3?
- Draw a line whose slope is the average rate of change from 4 to 2
- Where in the graph is the car slowing down?
- Where in the graph is the car speeding up?

Chapter 18- Derivatives of Functions

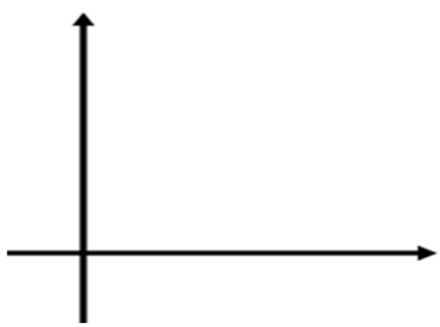
In the previous chapter we saw the idea of rates of change…

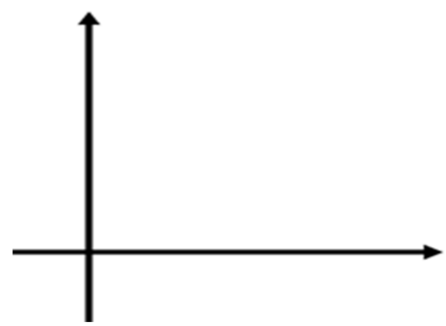
- Over a period: Average rate of change
- At a specific point: Instantaneous rate of change

$f'(b)$ is known as the derivative of $f(x)$ at the point $x = b$.

$f'(b)$ is a numerical value that denotes the instantaneous rate of change of the function $f(x)$ at the point $x = b$.

With the above in mind, $f'(b)$ tells us how the graph of the function is changing at the point $x = b$.
That is, $f'(b)$ tells us the slope of the line that is tangent to the point $f(b)$





We learned in chapter 17 that $f'(b) = \lim_{x \to b} \frac{f(x)-f(b)}{x-b}$

Similar to $f'(b)$ is the function $f'(x)$ that would let us know the instantaneous rate of change of $f(x)$ at any point by simply plugging in the value into the function.

In chapter 18 we discover that $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

If we let $x = b$ in it $f'(x)$ yields

Notation: The derivative of $y = f(x)$ is denoted by any of the following:

$$\frac{d}{dx}f(x) \qquad \frac{d}{dx}\,y \qquad \frac{dy}{dx} \qquad y' \qquad f'(x)$$

When computing $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ we need to know how to plug in $f(x+h)$, $f(x)$ and h.

$f(x)$ is simply the function itself and h is a new variable.

However, people often times get confused what to put in for $f(x+h)$ so what follow is an explanation of that term:

We now know that $f'(b) = \lim_{x \to b} \frac{f(x)-f(b)}{x-b}$

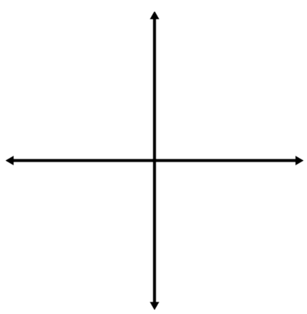
We learned previously that in for any limit to exist we need the right hand limit to equal the left hand limit.

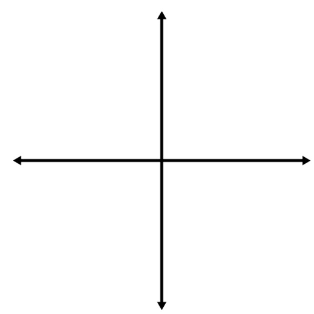
Thus, in this case, in order for $f'(x)$ to differentiable at the point $x = b$ we need

$$\lim_{x \to b^+} \frac{f(x) - f(b)}{x - b} = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$$

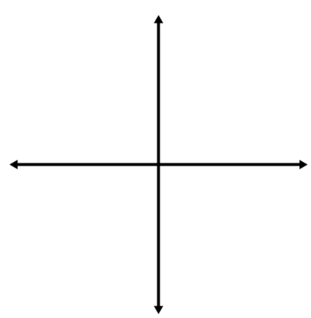
This limit does not always exist as illustrated by the following examples:

1) Both limits exist but are not equal

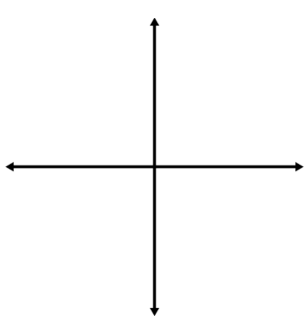




2) The value $f(a)$ does not exist



3) $\lim_{x \to a} f(x)$ DNE



Chapter 19- Computing Derivatives

Throughout chapters 17 & 18 we learned about derivatives (rates of change) and how to calculate the derivative of $f(x)$ using the limit definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

We can use the above definition to derive the following derivative rules:

Let C, b, m, n and a be constant real numbers

#	Name of Rule	Form of Function	Derivative
1	Power Rule	$f(x) = x^n$	$f'(x) = nx^{n-1}$
2	Sum Rule	$S(x) = f(x) + g(x)$	$S'(x) = f'(x) + g'(x)$
3	Constant Function	$f(x) = C$	$f'(x) = 0$
4	Constant Multiple	$f(x) = C \cdot g(x)$	$f'(x) = C \cdot g'(x)$
5	Linear Function (a combination of 1,2,3 & 4)	$f(x) = mx + b$	$f'(x) = m$
6	Product Rule	$P(x) = f(x) \cdot g(x)$	$P'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
7	Quotient Rule	$Q(x) = \frac{f(x)}{g(x)}$	$Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
8	Chain Rule	$C(x) = f(g(x))$	$C'(x) = f'(g(x)) \cdot g'(x)$
9	Exponential Function	$f(x) = e^{g(x)}$ $f(x) = a^X$	$f'(x) = g'(x) \cdot e^{g(x)}$ $f'(x) = \ln(a) \cdot a^X$

10	Sine Function	$f(x) = \sin(g(x))$	$f'(x) = g'(x) \cdot \cos(g(x))$
11	Cosine Function	$f(x) = \cos(g(x))$	$f'(x) = -g'(x) \cdot \sin(g(x))$
12	Logarithmic Function	$f(x) = \ln(g(x))$	$f'(x) = g'(x) \cdot \frac{1}{g(x)}$

Examples:

$f(x) = x^4 + 2x^3 - 16x + 9$

$f(x) = e^{2x^2}$

$h(y) = \cos(y^2)$

$h(t) = \sqrt{x} + \sqrt{x^3}$

$P(y) = (2y^3 - 6y)(18y - 8)$

$g(x) = \frac{\sin(x) + 2x^2}{x^3}$

$$w(t) = \ln(\frac{1}{2}t)$$

$$h(x) = \frac{1}{x^2} + \frac{1}{\sqrt{x}} - \sqrt[3]{x^2}$$

$$p(y) = \sin(2y)\, e^y + \cos(4y^2)$$

$$f(t) = \sin(2t^2 + t)$$

$$f(x) = x^9 e^{4x^2}$$

$p(t) = \ln(\frac{2t}{\sqrt{t}})$

Chain Rule Examples

$h(y) = (2y^3 + 16y)^3$

$m(x) = \sqrt{e^{2x} 8x^3}$

$p(y) = [\cos(y)\, e^y]^2$

Higher Order Derivatives

Thus far given the function $f(x)$ we have discussed finding the first derivative, $'(x)$.

We now turn to taking multiple derivatives of the same function and discuss why we might do so.

Let $f(x)$ be a function with first derivative $f'(x)$

Then $f''(x) = \frac{d}{dx}f'(x) = (f'(x))'$

And $\frac{d}{dx}f''(x) = f'''(x)$

Note that all the derivative rules we previously discussed still apply to higher order derivatives.

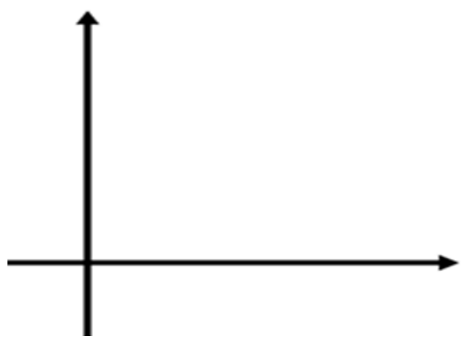
Meaning of Higher Order Derivatives

Let $f(x)$ be a function

Ex:

A line that is tangent to a curve is one that touches a single point without crossing through the shape. With this in mind a line tangent to a function represents the steepness of the function at that point.

Finding the Equation of a Tangent Line to $f(x)$ at the point $f(a)$



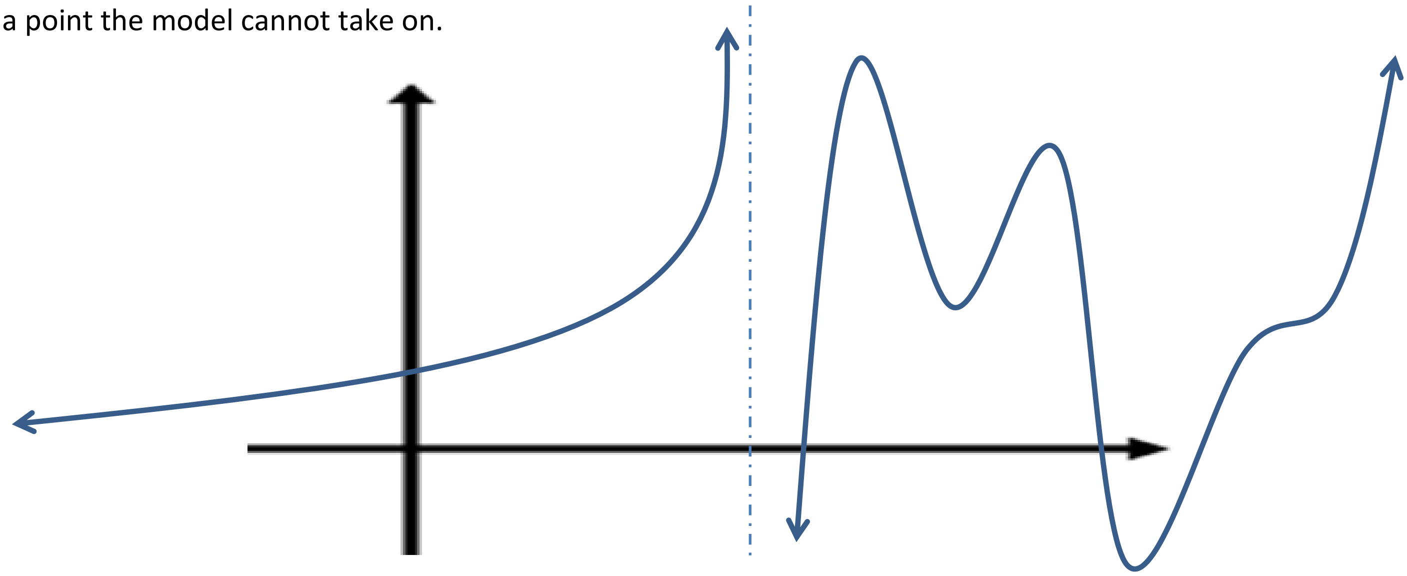
Procedure:

1) Find $f'(x)$
2) Calculate $f'(a)$
3) Use point $(a, f(a))$ to write the equation of the tangent line at $f(a)$

Ex: $g(x) = 2xe^{x^2}$. Find the equation of the tangent line to $g(1)$

Chapter 20- Finding and Classifying Critical Points

When considering the model of a biological process some of the more important values are the extreme points and discontinuities. Extreme points are where the process takes on maximum/minimum values while discontinuities are where the process does not exists mathematically. We call these critical points. Such points are important because extreme points could represent the highest/lowest value of a population in the wild, or the concentration of a drug in the blood steam or perhaps the minimum density of a disease traveling through a population. A discontinuity could exist at a point where the model does not make sense to exist. For example, rational models (fractional functions) usually have a value that makes the denominator equal zero at a point the model cannot take on.



Consider the above curve of some continuous function $f(x)$

We say that $f(x)$ has a Local Maximum/Minimum at $x = a$ if $f(x)$ is less than/greater than $f(a)$ at points surrounding $f(a)$

We say that $f(x)$ has a Global Maximum/Minimum if $f(x)$ attains its largest/smallest value at $f(a)$.

- In order for a point $x = a$ t to be a maximum, the graph needs to be increasing before $x = a$ and decreasing after $x = a$

- In order for a point $x = a$ t to be a minimum, the graph needs to be decreasing before $x = a$ and increasing after $x = a$
    - In either case above, the sign of the derivative switches at $x = a$
    - This implies that the function has no change at $x = a$

- A point $x = a$ is a point of discontinuity if $f(a)$ does not exist (DNE). (Often $f'(a)$ DNE either).
- Sometimes $f'(a) = 0$ but the point is not a max nor a min.
    - We call these points of inflection (see graph above)
    - This is where a curve changes concavity (defined later)

Considering the information on the previous page, we find critical points by:

A)

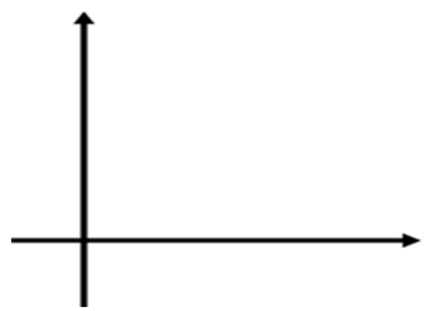
B)

It is clear that a point that results from method B) is a discontinuity. However, once we find critical points from method A) we need a way to determine whether they are maximums, minimums or points of inflection.

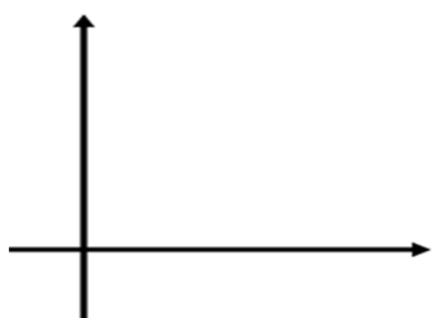
The First Derivative Test

Let $f(x)$ be a continuous function, $a < b < c$ and suppose that $f'(b) = 0$.
Since $f'(b) = 0$, there must be a critical point at $f(b)$. To classify it we can use a sign chart and the first derivative $f'(x)$.

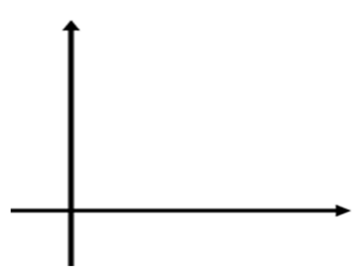
X value	$f'(x)$
a	$f'(a) > 0$
b	$f'(b) = 0$
c	$f'(c) < 0$

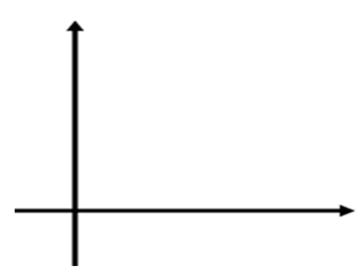


X value	$f'(x)$
a	$f'(a) < 0$
b	$f'(b) = 0$
c	$f'(c) > 0$



X value	$f'(x)$
a	$f'(a) < 0$ or > 0
b	$f'(b) = 0$
c	$f'(c) < 0$ or > 0





Examples:

Let $y = 4x^2 + 6x$. Find all critical point(s) and classify them using the first derivative test.

Let $g(y) = y^3 - 10$. Find all critical point(s) and classify them using the first derivative test.

Let $h(t) = t^3 + t^2 - t + 1$. Find all critical point(s) and classify them using the first derivative test.

Concavity

The concept of concavity of a function is best grasped using pictures:

Concave Up **Concave Down**

Another approach…

We know that $f'(x) = \text{the rate of change of function } f \text{ at point } x$

Then $\left(f'(x)\right)' = f''(x) = \text{the rate of change of the rate of change.}$ That is, $f''(x)$ tell us how the rate of change is changing.

So if $f''(a) < 0$ this implies that $f'(x)$ is decreasing at $x = a$,which means that the graph is flattening out:

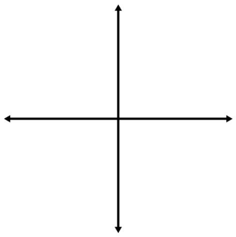
Similarly, if $f''(a) > 0$ then the graph is becoming more steep at $f(a)$:

Based on the above we state the following facts:

- If $f''(a) < 0$ then $f(x)$ is <u>concave down</u> at $x = a$
- If $f''(a) > 0$ then $f(x)$ is <u>concave up</u> at $x = a$
- If $f''(a) = 0$ then $f(x)$ is neither concave down nor up at $x = a$
  - Instead this is almost always a point of inflection (there are a few special cases where this is not true)
  - A <u>Point of Inflection</u> is a place where the graph switches concavity
  - If $f''(a) = 0$ in any function we will encounter, it will be a point of inflection

<u>Ex:</u> $g(x) = x^3 - x + 1$

Ex: Let $f(x) = x^3 + x^2 - x + 1$.
Determine the concavity of $f(x)$ at $x = -3$ and at $x = 0$. Where does $f(x)$ change concavity?



Recall that since max/mins are "flat" points on a graph, we solve $f'(x) = 0$ in order to find them.
So far we used $f'(x)$ to help classify critical points. We can also use $f''(x)$ to determine concavity at these values in order to classify them.

The Second Derivative Test

Suppose $f'(c) = 0$

- If $f''(c) > 0$ then $f(c)$ is a min
- If $f''(c) < 0$ then $f(c)$ is a max
- If $f''(c) = 0$ then $f(c)$ is a point of inflection

Examples

Let $h(t) = t^3 + t^2 - t + 1$. Find all critical point(s) and classify them using the second derivative test.
We have seen this problem previously so I won't spend time finding the critical points.

Let $g(x) = x + \frac{9}{x}$. Find all critical point(s) and classify them using the second derivative test.

Let $g(x) = 1 + x^2 - \frac{x^6}{3}$. Find all critical point(s) and classify them using the second derivative test. Also, determine all values where $f(x)$ is concave up and concave down.

Finding the max/mins of a function on a closed interval

More often than not, Biological models have restricted domains and/or ranges.

For example:

- $M(w)$ which tells us the milligrams of medicine a person who weighs weight w requires

- Any time based model (ie, $f(t)$ where t is time)

- Probability function $P(x)$ (from 151)

Procedure for finding the max/mins of function $f(x)$ on a closed interval $[a, b]$:

1) Find all critical points of $f(x)$ as discussed previously
2) Eliminate any that are not found in the interval $[a, b]$
3) Plug all critical points <u>and end points of the interval</u> in the function to see which is the largest/smallest

<u>Ex:</u> Let $f(x) = x^4 - 2x^2$. Find the maximum and minimum of $f(x)$ on the interval $[-1, 2]$.

Lets practice analyzing a bit more complicated function.

Let $f(x) = \frac{x^2+2x+1}{x+2}$

Sketching $f(x)$ given information about $f'(x)$

Since $f'(x)$ tells us where $f(x)$ is increasing/decreasing and where the maximums/minimums are, we can use this information to sketch the general shape of $f(x)$.

Ex: Suppose we used the 1st derivative test and obtained the following sign chart:

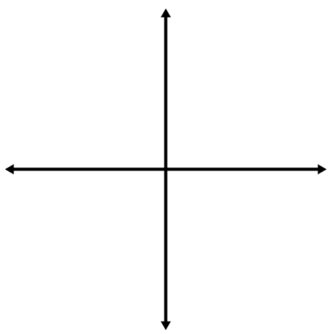
Ex

Ex (assume the discontinuities are infinite discontinuities)

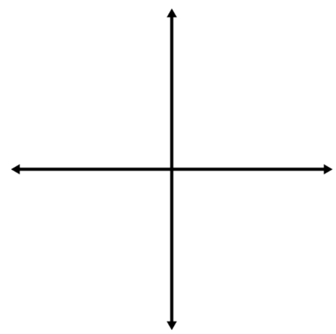
Exponential Models

An exponential model is a model containing the term $e^{\alpha t}$ where α is a constant and t is a variable.

1) If $\alpha > 0$



2) If $\alpha < 0$



We can use exponential models to describe situations where there is either growth ($\alpha > 0$) or decay ($\alpha < 0$) in unlimited environments.

A Simple Example:

Suppose $N(t)$ tells us the population of a type of animal at time t. If a population lives in an area that contains unlimited resources then the population will grow uncontrollably.

More specifically, the growth of the population would be proportional to the number of individuals present.

That is, $N'(t) = k \cdot N(t)$

where $N(t) = \text{the population}$ at time t and $k = \text{the birthrate of the population.}$

This differential equation can be solved using methods found in chapters 26-28 to obtain the following solution:

$$N(t) = N_0 \cdot e^{kt}$$

where $N_0 = \text{the initial population}$ and $k = \text{the birthrate of the population.}$

If $k > 0$ then the population is growing

If $k < 0$ then the population is declining

Optimization Problems

Getting the most out of a situation is something everyone strives for. This is especially the case when considering complex systems that contain objectives and constraints whose relationship with each other have unclear tradeoffs and implications. Such systems arise in countless applications in sciences, business and everyday life.

- What levels of light and/or nutrients yield the best crops?
- What is the best way to get our product to the distributors?
- Given a limited quantity of supplies, how can we make the most effective or most cost efficient or largest product?
- What dosage/frequency of a drug will effectively fight the disease without hurting the patient or resulting in the patient becoming broke?
- What is the least amount of effort I can put into math 152 while still getting the grade I want?
- How can we employ our resources to limit the population of an invasive pest while efficiently managing our funds?

Since a major application of calculus is finding the maximums/minimums of functions, it provides us a tool in which to answer some of these types of questions so that we can achieve the optimal result.

All optimization problems are different (which is fun and exciting!)

What follows are general guidelines to solving optimization problems:

1) Define the key variables in the problem
    - There can be many variables in real world applications
    - We will generally only see 2 in each problem

2) Draw a picture relating the variables (if applicable)

3) Derive the objective function $Q(x)$
    - This is the function that we are trying to maximize/minimize
    - This function should be written in terms of the variables found in 1)

4) Determine if there exist any constraints on the variables

5) Write $Q(x)$ in terms of only 1 variable
    - Usually requires solving for one variable and substituting it into the other

6) Use calculus to find the max/min of $Q(x)$

7) Make sure your answer is reasonable
    - Help catch mistakes and/or make improvements

Optimization Examples:

Suppose the function $Y(N) = \frac{N}{1+N^2}$ models the yield Y of a crop given the nitrogen level N.

What Nitrogen level yields the most crops?

Suppose $C(t)$ tells use the concentration of a drug in the blood stream t hours after injecting it as given by

$$C(t) = \frac{6t}{2t^2 + 1}$$

What is the highest concentration the drug will achieve in the blood steam?

More Optimization

A farmer has a grazing area and has 3000 feet of fencing to make a rectangular pen. The grazing area is next to a river so the farmer only needs to construct 3 walls. What height and width of the pen would maximize the area of the pen?

More Optimization

Find two nonnegative numbers whose sum is 9 and that maximizes the product of one with the square the other.

More Optimization

The average individual daily milk consumption for Charolais, Angus and Hereford calves is approximated by the function

$$M(t) = 6.281t^{0.242}e^{-0.025t} \quad , \qquad 1 \leq t \leq 26$$

Where $M(t)$ is the milk consumption (in kg) and t is the age of the calf (in weeks).
Find the age of a calf at which maximum daily consumption occurs.
How much milk is consumed on this day?
Do you expect this value to be exactly the same for all calves?

Chapter 21- Estimating the Area Under a Curve

Key Question: How can we get some sort of cumulative/total type number from a function?

For Example, Suppose $f(x)$ gives the number of births taking place in a town x minutes after the year 2000 and we want to know the total number of births from x= a to $x = b$.

The answer to this is the number of births at each instance added up over the entire time period. In other words, the exact answer would be area between the curve of $f(x)$ and the x-axis.

Key Idea: One way to estimate this answer would be to find out how many births are taking place between small intervals, assume the same number of births took place between and then adding up each number of births times how long went by. This can be viewed as taking an average rate of change at various times and summing them.

How can we can do this:

1) Break up the length of time into n equal sized pieces

$$chnage\ in\ time = \Delta x = \frac{b-a}{n}$$

2) Let $x_0 = a, x_1 = a + \Delta x,\ x_2 = a + 2\Delta x, \dots\dots\dots.., x_n = b$

3) Add up each function value at each x_i and multiply it by the change in time, Δx. Note that you need to either overestimate your answer or underestimate your answer which leads to two different approximations:

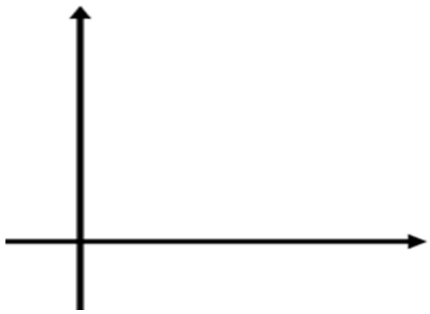
$$the\ area\ under\ the\ curve \approx \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x \quad \text{or} \quad \approx \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

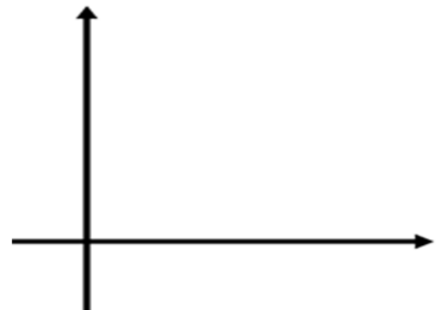
Example: Let $f(x) = x^2 + 1$. Estimate the area under the curve from $x = 0$ to $x = 5$ using both an upper and lower sum. Break the interval up into 10 pieces, that is, let $n = 10$.

Chapter 22- Antiderivatives and The Fundamental Theorem of Calculus

The area located between a function and the x-axis between two points provides us with the total output given by the function between those points. In chapter 21 we saw the idea of how to approximate this area using rectangles.

From Ch 21:

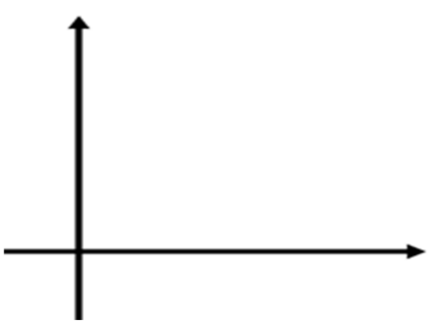




Integral of a Function

$$\text{The exact area between } f(x) \text{ and the x–axis between } x=a \text{ and } x=b = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x$$

Here (Ch 22):



Antiderivatives

The antiderivative of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$

Ex: Suppose $f(x) = 2x$. What is $F(x)$?

Family of Antiderivatives

Notation: $\int f(x)\ dx = \ F(x) + C$

Ex: Let $g(y) = x^2 + 2x + 1$. Find the antiderivative of $f(x)$.

Examples:

$$\int 4x + 2\, dx$$

$$\int x^3 + \sqrt{x} + \frac{1}{x^2}\, dx$$

$$\int \sin(4x)\, dx$$

$$\int e^{4t}\, dt$$

From last class, the antiderivative of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. Now that we know how to find an antiderivative we can discuss…

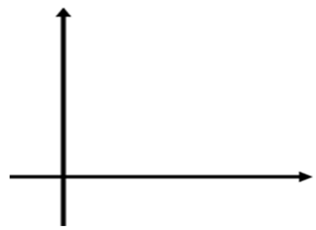
The Fundamental Theorem of Calculus

- If $f(x)$ is continuous on the interval $[a, b]$, then the function $F(x) = \int_a^x f(x)\ dx$ is continuous on $[a, b]$, is differentiable on (a, b) and $F(x) = f'(x)$.

- Furthermore, the area between $f(x)$and the $\text{x} - \text{axis is given by}$

$$\int_a^b f(x)\ dx = F(b) - F(a) \text{ where } F(x) \text{ is the antiderivative of } f(x).$$

Notation: $F(x)\Big|_a^b = F(b) - F(a)$



Question: What about the "$+C$" ?

Area above the x-axis is positive area. For example consider $\int_1^1 e^{2x}\ dx$.

Area below the x-axis is <u>negative area.</u> For example, consider $\int_0^2 t^2 - 4t \ dt$.

If a graph has area both above and below the x-axis then the integral sums the total of each piece and gets a cumulative result as illustrated in the following two examples:

<u>Ex:</u> $\int_0^{2\pi} \sin(y) \ dy$

<u>Ex:</u> $\int_{-1}^{4} \mathrm{x}^3 \ dx$

We have introduced two different types of integrals and it needs to be clear the differences between the two:

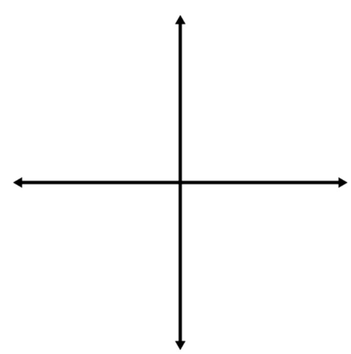
1) Indefinite Integrals

$$\int f(x)\ dx = F(x) + C$$

2) Definite Integrals

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

For Indefinite Integrals, how do we find the "$+C$"?



Ex: Suppose $P'(t) = 300e^{2t}$ represents the instantaneous rate of change of a population at time t. Suppose we know there were 300 individuals to start (ie, at $t = 0$).
Find the exact equation for $P(t)$.

Averages

We denote the average value of $f(x)$ between $x = a$ and $x = b$ by $\overline{f}$ where

$$\overline{f} = \frac{\text{total function values between a and b}}{\text{distance from a to b}} = \frac{\int_a^b f(x)\ dx}{b-a} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

Ex: Let $N(t) = 1000e^{2t}$ be the population of ants on The Hill at UT t years after this moment. Determine the average population size over the next two years.

Ex: Suppose $f(x) = 71.3x - 4.15x^2 + .434x^3$ tells us the force f (in Newtons) exerted by a tendon as it is stretched x millimeters. Determine the average force exerted between 2mm and 11mm.

Chapter 23- Methods of Integration

In chapter 22 we first learned how to find the antiderivative of a function, known as an indefinite integral: $\int f(x)\ dx = F(x) + C$.

We then saw how to use the antiderivative to evaluate the area under a curve. This is known as a definite integral: $\int_a^b f(x)\ dx = F(b) - F(a)$.

For more complicated functions, the antiderivative does not fit into one of our previously discussed forms.

To handle these antiderivatives we have:

A) The Substitution Method

And

B) Integration By Parts

The Substitution Method

Recall the Chain Rule: $\frac{d}{dx} f\big(g(x)\big) = f'(g(x)) \cdot g'(x)$

Let $h(x)$ be a function that appears to be the result of a chain rule: $h(x) = f'\big(g(x)\big) \cdot g'(x)$.

Here we want to find the antiderivative of $h(x)$: $\int h(x)\, dx = \int f'(g(x)) \cdot g'(x)\, dx$.

We can use the substitution method to find the antiderivative of $h(x)$ using the following steps:

1) Let $u = g(x)$ where $g(x)$ is usually a function that is plugged into another function
2) Take the derivative of $u = g(x)$. That is, find $\frac{du}{dx} = g'(x)$.
3) Solve for dx to obtain $dx = du \cdot \frac{1}{g'(x)}$
4) Substitute $u = g(x)$ and $dx = du \cdot \frac{1}{g'(x)}$ into $\int f'(g(x)) \cdot g'(x)\, dx$ to obtain $\int f'(u)\, du$
5) Integrate: $\int f'(u) du \quad = \quad F(u) + c$
6) Substitute $u = g(x)$ back in: $F(u) + c \quad = \quad F\big(g(x)\big) + c$

Substitution Examples

$$\int (1 + x^3)^4 \quad dx$$

$$\int \frac{1}{(12 - 5y)^2} \; dy$$

Substitution Examples

$$\int \frac{\mathrm{x} \cdot \ln(1+x^2)}{1+x^2} \quad dx$$

$$\int \cos(x)\; e^{\sin(x)}$$

So far we have only seen how to evaluate an indefinite integral: $\int f'(g(x)) \cdot g'(x)\, dx = F(g(x) + C$

If we want to compute a definite integral $\int_a^b f'(g(x)) \cdot g'(x) \quad dx$ we have two options:

1) Replace the original bounds:

2) Substitute $u = g(x)$ back in:

Using a previous example:

$$\int_1^5 (1 + x^3)^4 \cdot x^2 \quad dx$$

Another definite integral example:

$$\int_{\frac{\pi}{2}}^{\pi} 2x \cdot \cos(x^2) \quad dx$$

Definite integral using a previous example:

$$\int_{0}^{2} \frac{1}{(12-5y)^2} \; dy$$

Integration By Parts

Recall the Product Rule: $\frac{d}{dx}\left[\, f(x) \cdot g(x) \,\right] = f'(x) \cdot g(x) \;+\; f(x) \cdot g'(x)$

Here we want to find the antiderivative of the right hand side, so we integrate both sides:

$$\int \frac{d}{dx}[f(x) \cdot g(x)] \; dx \;=\; \int f'(x) \cdot g(x) + f(x) \cdot g'(x) \; dx$$

Finally, we obtain the following result:

$$\int f(x) \cdot g'(x) \;=\; f(x) \cdot g(x) - \int f'(x) \cdot g(x)$$

Sometimes represented as: $\int \; u \cdot \, dv \;=\; u \cdot v \;-\; \int \; du \cdot v$

How To Use Integration By Parts

- Given $\int f(x) \cdot g(x) \; dx$ you need to choose one of $f(x)$ or $g'(x)$ to be "u" and the other will be "dv".
- We will differentiate "u" so choose "u" to be the one that taking the derivative makes it "less nasty"
    - Here is an acronym to help you decide what "u" should be. Then, "dv" is whatever is left over.
    - Let u be the first of the following to appear in the integral:
        - I . L . A . T . E .
- Differentiate u to obtain du and Integrate dv to obtain v
- Plug u, du, dv and v into the formula $\int \; u \cdot \, dv \;=\; u \cdot v \;-\; \int \; du \cdot v$
- Evaluate $\int \; du \cdot v$ and simplify

Examples Using Integration By Parts

$$\int e^{x} \cdot x \quad dx$$

$$\int \sin(t) \cdot 2t \quad dt$$

For a <u>definite</u> integral using integration by parts, use the following formula:

$$\int_a^b u \cdot dv = \left. u \cdot v \right|_a^b - \int_a^b v \cdot du$$

Examples

$$\int_1^6 e^x \cdot x \; dx$$

$$\int_5^{10} ln\,(x)\; dx$$

More examples of integration by parts

$$\int_{0}^{10} \sin(t) \cdot 2t \quad dt$$

Sometimes you need to do integration by parts twice:

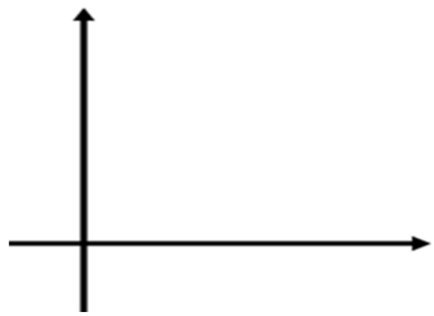
$$\int \cos(x) \cdot (2x^2 + 3x) \; dx$$

Chapter 24- Applications of Integration

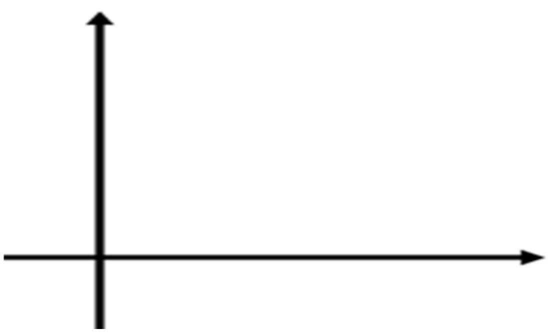
Much of the research being done in modern science makes use of the power of calculus to determine extreme values and find total amounts. This section aims to illustrate some of the applications of integration specifically, but please understand that there are countless more uses for integrals that will not be explored here.

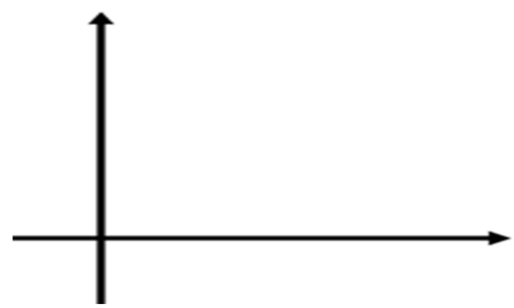
Finding the Area Between Two Curves

Suppose we want to find the area between two functions $f(x)$ and $g(x)$.



What we really want to do is find the area under the upper curve and then subtract off the area under the lower curve:





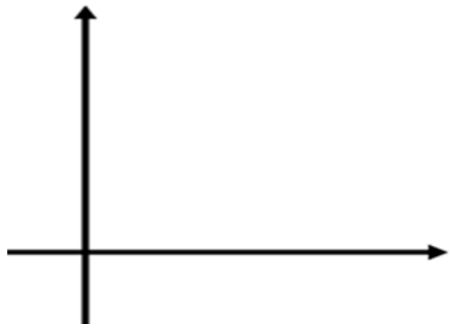
With the above in mind, the area, A, between two curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is given by:

$$A = \int_a^b g(x)\ dx - \int_a^b f(x)\ dx = \int_a^b (g(x) - f(x))\ dx$$

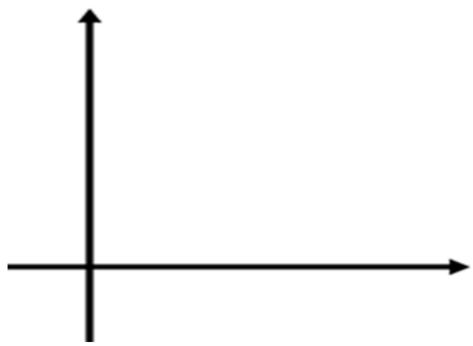
Where $g(x) \geq f(x)$ on $[a, b]$

Examples

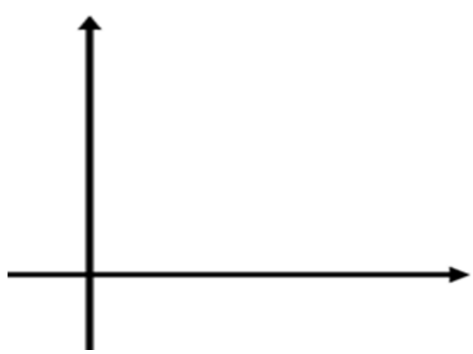
Find the area bounded between $x^2 + 10$ from $x = 0$ to $x = 2$



Find the area bounded between $x^2 + 10$ from $x = 2$ to $x = 10$



Find the area bounded by $\ln(x)$, $\frac{5}{7}x - 5$ and the x-axis.



Remember that when we compute $\int_a^b \cdot \; dx$, we are calculating total amounts between $x = a$ and $x = b$.

Therefore $\int_a^b f(x) \; dx$ tells the sum of all function values between $x = a$ and $x = b$.

If the function is a rate of change (ie $f = f'(x)$), then $\int_a^b f'(x)\,dx$ will tell us the total change of $f(x)$ between $x = a$ and $x = b$.

The following examples illustrate the value of an integral.

Example:

Pollution enters a lake at $t = 0$ given by the formula $P'(t) = 10(1 - e^{-0.5t})$ where t is measured in hours and $P'(t)$ is a rate of change with units $\frac{\text{gallons}}{\text{hour}}$.

At the same time a filter removes pollution at a rate given by $f'(t) = 0.4t$ with the same units as $P'(t)$.

How much pollution exists in the lake after 12 hours?