5.3- Trigonometric Graphs (Sine and Cosine)

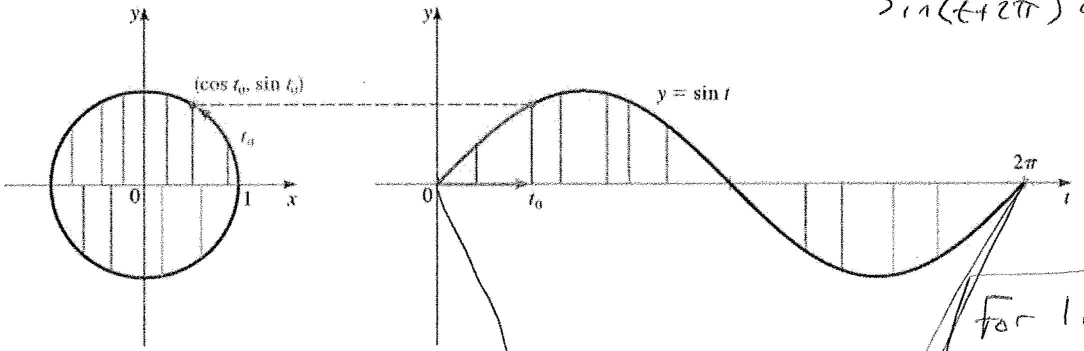
This section focuses on graphs of **sine** and **cosine** functions. Others are in the next section

PERIODIC PROPERTIES OF SINE AND COSINE

The functions sine and cosine have period 2π:

$$\sin(t + 2\pi) = \sin t \qquad \cos(t + 2\pi) = \cos t$$

Meaning: Period – How long it takes the graph (y-values) to repeat itself since the unit circle has circumference 2π the (x,y) values repeat every 2π radians. Thus $\sin t$ and $\cos t$ have same values as $\sin(t+2\pi)$ and $\cos(t+2\pi)$



For 1 period of $\sin t$
- min/max at ± 1
- min occurs at $\frac{3\pi}{2}$
  max at $\pi/2$
- x-intercepts: $x=0$
  $x=\pi$
  $x=2\pi$

General forms of Sine and Cosine

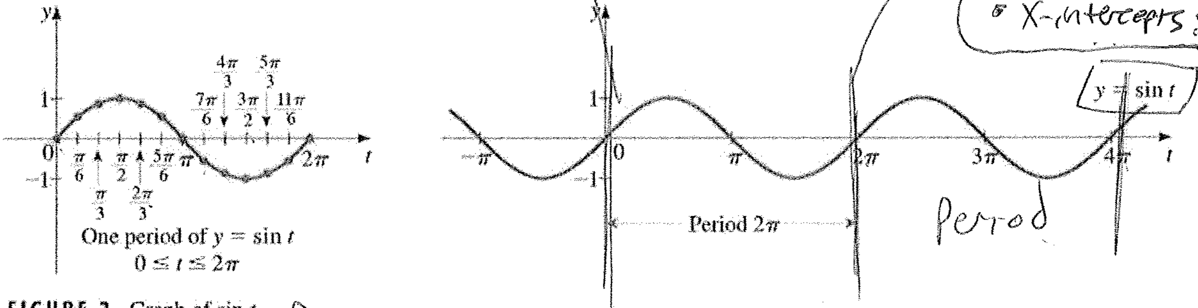


FIGURE 2 Graph of sin t

Reference these when drawing transformations of sin/cos

Cos t

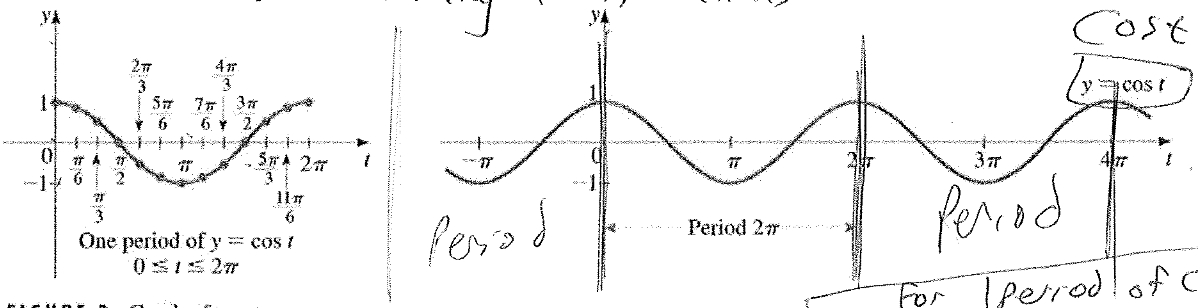
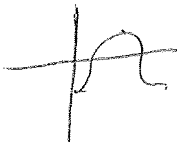


FIGURE 3 Graph of cos t



For 1 period of $\cos t$
- min/max at ± 1
- min occurs at π
  max at 0 and 2π
- x-intercepts at $x=\pi/2$, $x=\frac{3}{2}\pi$

Graphs of Transformations of Sine and Cosine

We can use the techniques found in section 2.5 paired with the general forms of Sine and Cosine seen above (pg 1) to draw transformations of the graphs sin *t* and cos *t*.

"c" shifts the graph up/down

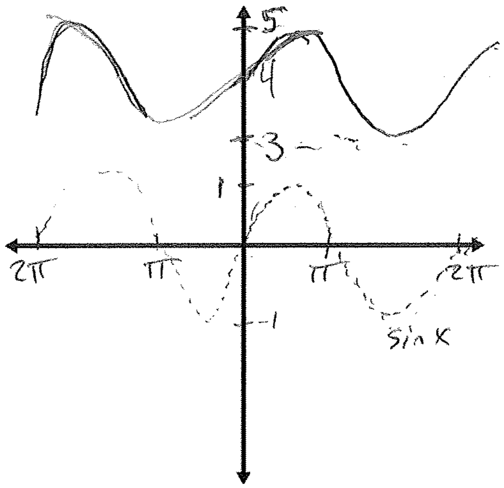
Form 1: $y = c + a\sin x$ or $y = c + a\cos x$

Explanation of form: "a" stretch/shrink graph. Since sin x, cos x have min/max at ±1, c + a sin x and c + a cos x have min/max values at $\boxed{c \pm a}$

|a| is known as the amplitude which is the vertical height of the graph

$y = 4 + \sin x$

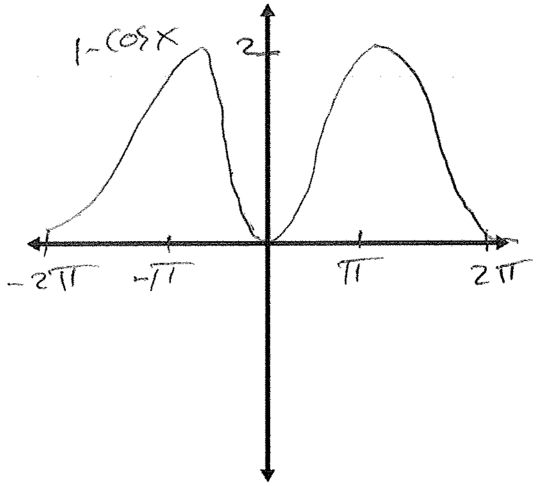
max/min at $c \pm a = 4 \pm 1 = 3, 5$

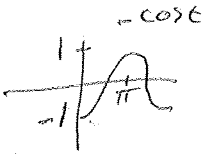


$y = 1 - \cos x$

shift up 1 unit

flip upside down





max/min at $c \pm a$

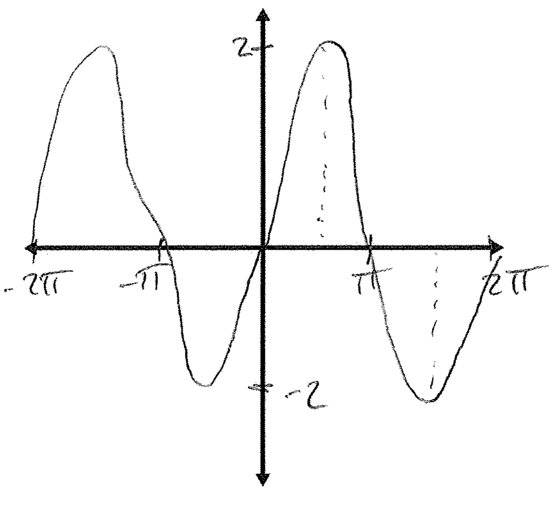
$= 1 \pm -1$

$= 0, 2$

$y = 2\sin x$

↳ stretch sin x by factor of 2

min/max $0 \pm 2 = -2, 2$

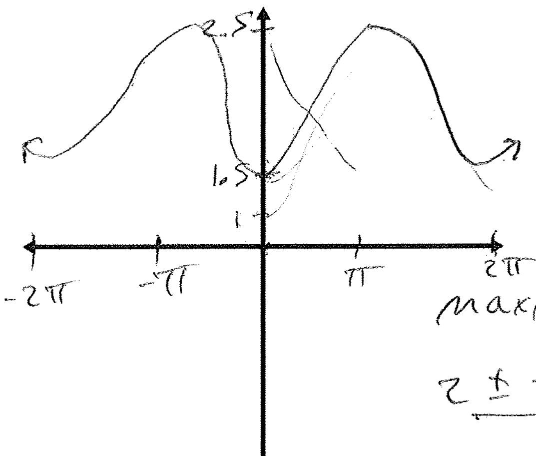


$y = 2 - \frac{1}{2}\cos x$

shift up 2

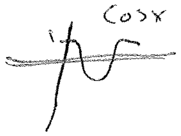
↳ flip upside down

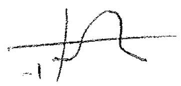
shrink by 1/2



max/min

$2 \pm -\frac{1}{2} = 2.5/1$





f(x) then f(kx) is a horizontal stretch

Form 2: $y = c + a\sin(kx)$ or $y = c + a\cos kx$

SINE AND COSINE CURVES

The sine and cosine curves

$$y = a\sin kx \quad \text{and} \quad y = a\cos kx \quad (k > 0)$$

have **amplitude** $|a|$ and **period** $2\pi/k$.

An appropriate interval on which to graph one complete period is $\boxed{[0, 2\pi/k]}$

Amplitude- the height of the graph

Period- How long it takes for the function to cycle through its y-values. That is, how long until the function starts to repeat its y-values

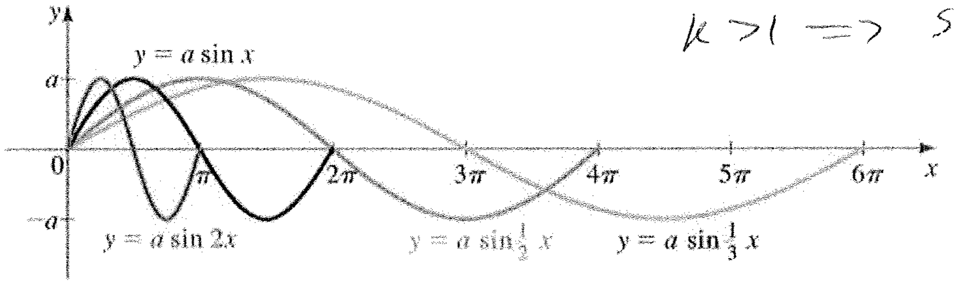
Explanation of form: "a" and "c" act the same as previous page

the difference here is that "k" determines how often to repeat the graph. On pg 1 graphed on $[0, 2\pi]$ Now graph on $[0, \frac{2\pi}{k}]$

Visual Example using $y = a\sin x$:

$0 < k < 1 \Rightarrow$ stretch horizontally

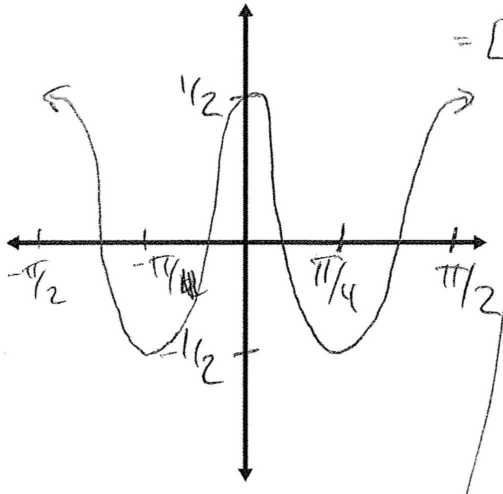
$k > 1 \Rightarrow$ shrink horizontally

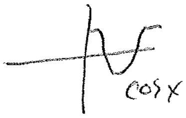


$y = \frac{1}{2}\cos 4x$

$a = \frac{1}{2}$, $c = 0$ so max/min at $0 \pm \frac{1}{2} = -\frac{1}{2}, \frac{1}{2}$

$k = 4$ so period $= [0, \frac{2\pi}{k}] = [0, \frac{2\pi}{4}]$
$= [0, \frac{\pi}{2}]$



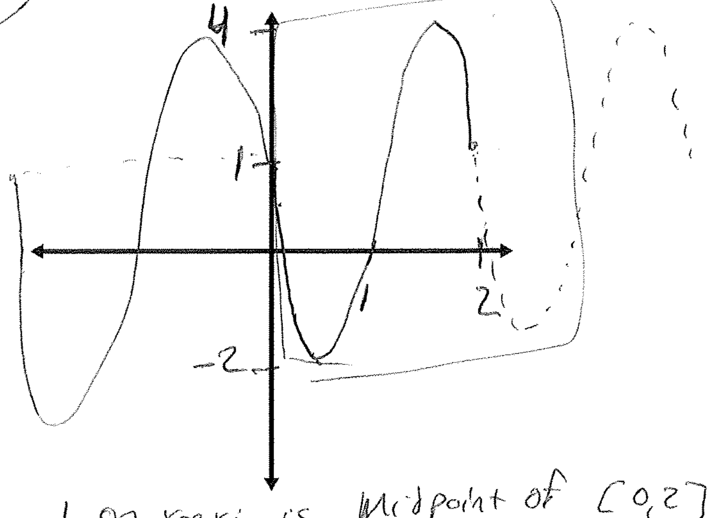


Flip sin x upside down

$y = 1 - 3\sin \pi x$

$a = -3$, $c = 1 \Rightarrow$ max/min at $1 \pm -3 = -2, 4$

$k = \pi$ so period $= [0, \frac{2\pi}{\pi}] = [0, 2]$



1 on x-axis is midpoint of $[0, 2]$

$\boxed{\text{1 on y-axis is midpoint of } [-2, 4]}$

Form 3: $y = c + a\sin k(x - b)$ or $y = c + a\cos k(x - b)$

SHIFTED SINE AND COSINE CURVES

The sine and cosine curves

$$y = a\sin k(x - b) \quad \text{and} \quad y = a\cos k(x - b) \quad (k > 0)$$

have **amplitude** $|a|$, **period** $2\pi/k$, and **phase shift** b.

An appropriate interval on which to graph one complete period is $\boxed{[b, b + (2\pi/k)]}$.

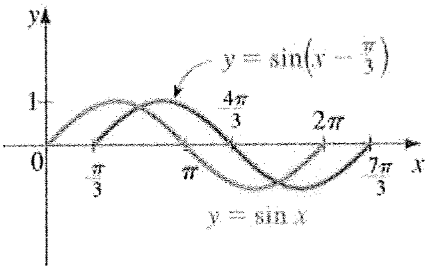
Amplitude- largest value that the function obtains

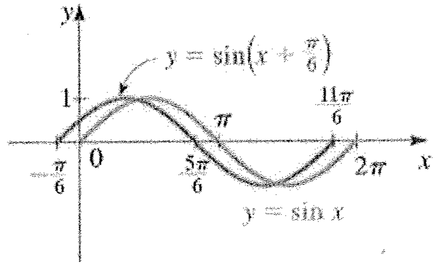
Period- How long it takes for the function to cycle through its y-values. That is, how long until the function starts to repeat its y-values

Phase Shift- a horizontal shift in the graph *b* units

Explanation of form: everything is same as previous page except you now need to plug "b" into the formula for the period which will shift the graph left/right

Visual Reference:





$b > 0 \rightarrow$ shift left b units

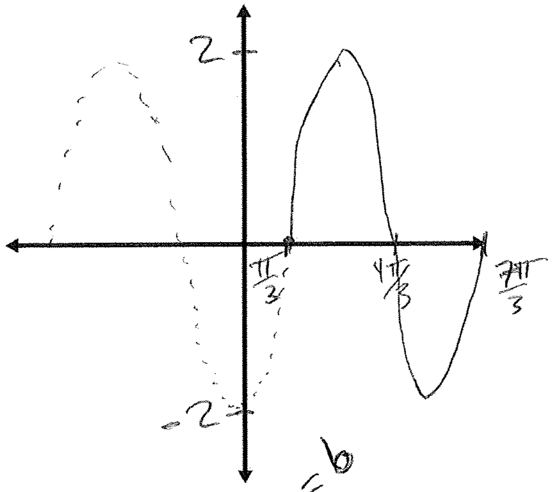
$b < 0$ shifts right b units

$c = 1, a = 1, k = 3, b = -\pi/6$

↑

$y = 2\sin(x - \frac{\pi}{3})$ c±a

- will have min/max at $0 \pm 2 = 2, -2$
- Amplitude $|2| = 2$
- period: $\frac{2\pi}{k} = \frac{2\pi}{1} = 2\pi$



- Phase Shift is $\overset{b}{\pi/3}$ so graph on

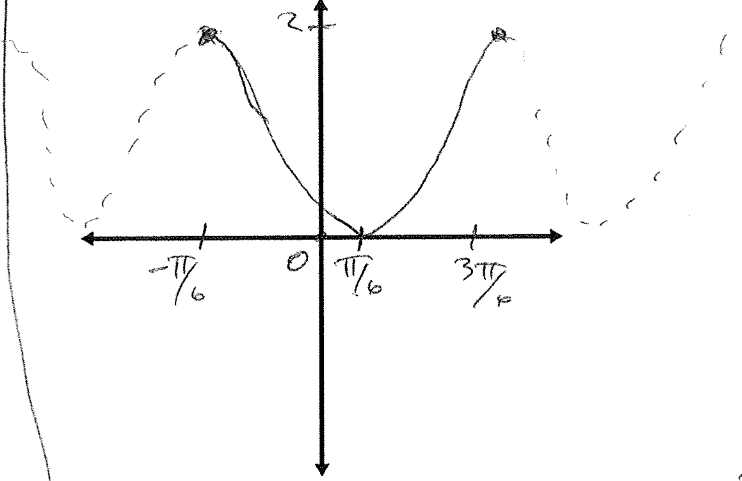
$[b, b + \frac{2\pi}{k}] = [\pi/3, \pi/3 + 2\pi] = [\pi/3, \frac{7\pi}{3}]$

$y = 1 + \cos(3x + \frac{\pi}{2}) = 1 + \cos(3(x + \frac{\pi}{6}))$

min/max $1 \pm 1 = 2, 0$

Amplitude $= |a| = |1| = 1$

Period $= \frac{2\pi}{3}$



Phase shift $\pi/6$ so graph on

$\pi/2$

$[b, b + \frac{2\pi}{k}] = [-\frac{\pi}{6}, -\frac{\pi}{6} + \frac{2\pi}{3}] = [-\frac{\pi}{6}, \frac{3\pi}{6}]$

5.4- More Trigonometric Graphs

Note: The vertical asymp. / x-intercepts

Graphs of Tangent, Cotangent, Secant and Cosecant

V.A.

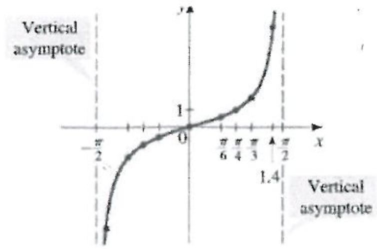


FIGURE 1
One period of $y = \tan x$

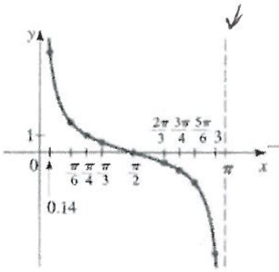


FIGURE 2
One period of $y = \cot x$

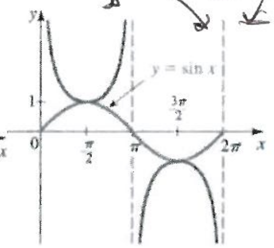


FIGURE 3
One period of $y = \csc x$

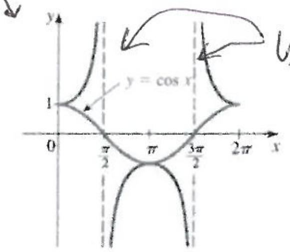


FIGURE 4
One period of $y = \sec x$

VA

PERIODIC PROPERTIES

The functions tangent and cotangent have period π:

$$\tan(x + \pi) = \tan x \qquad \cot(x + \pi) = \cot x$$

The functions cosecant and secant have period 2π:

$$\csc(x + 2\pi) = \csc x \qquad \sec(x + 2\pi) = \sec x$$

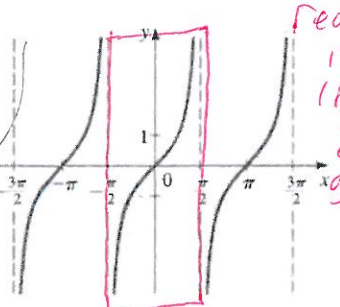
$\tan t = \frac{y}{x}$

$\cot = \frac{x}{y}$

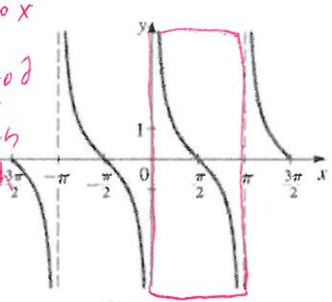
$\csc t = \frac{1}{y}$

$\sec t = \frac{1}{x}$

red box is 1 period of each graph

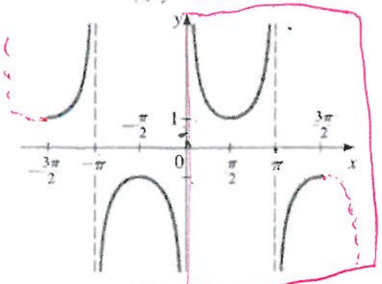


(a) $y = \tan x$

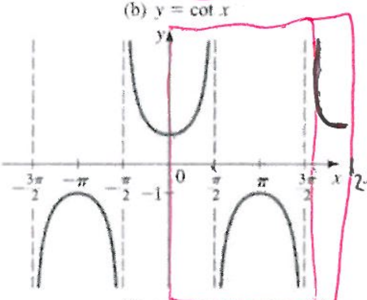


(b) $y = \cot x$

- For tan, cot, the ratios $\frac{y}{x}, \frac{x}{y}$ repeat every π units sothats why tan, cot have period
- For csc, sec, x and y values repeat every 2π units so thats why they period 2π.



(c) $y = \csc x$



(d) $y = \sec x$

- The vertical asymptotes in the graphs are also related to the functions form
  ↳ they occur where denominator = 0

Explanation:

- in Unit circle, $x=0$ for $t = \pi/2, \frac{3\pi}{2}$ which is why tan and Sec have vert. Asym. at $\pi/2$ and $3\pi/2$
- in unit circle, $y=0$ for $t=0, \pi, 2\pi$ which is why Cot, csc have vert Asym. at these values

need to graph
- Asymptotes
- X-intercept(S)
- General Form

Transformations of Tangent and Cotangent

TANGENT AND COTANGENT CURVES

The functions

$$y = a\tan kx \qquad \text{and} \qquad y = a\cot kx \qquad (k > 0)$$

have $\boxed{\text{period } \pi/k.}$

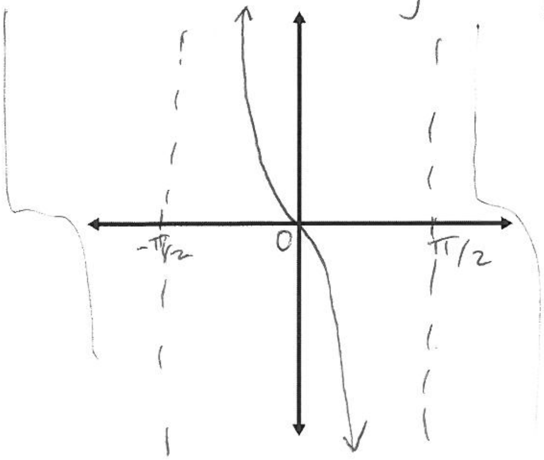
Using information from section 2.5 paired with the general forms seen above we can graph these functions

$y = -4\tan x$

-4 flips tan x upside down and stretches vertically

Period: $\frac{\pi}{k} = \frac{\pi}{1} = \pi$

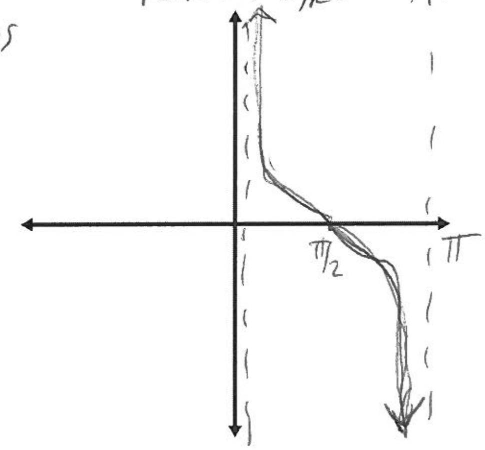
⇒ asymptotes in normal spots



$y = \frac{1}{2}\cot x$

$\frac{1}{2}$ flattens cot

period: $\pi/k = \pi/1 = \pi$ → normal asymptotes



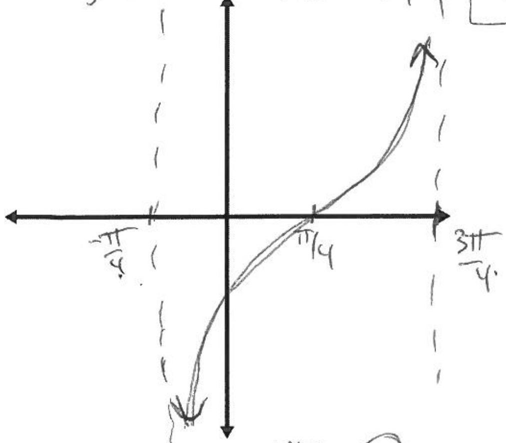
$y = \frac{1}{2}\tan\left(x - \frac{\pi}{4}\right)$

$\frac{1}{2}$ shrinks tan x by factor of 2

$-\pi/4$ shift tan x right $\pi/4$ units

So shift asymp. and intercept w/ it.

new asym. at $\pm\pi/2 + \pi/4 = \boxed{-\frac{\pi}{4}, \frac{3\pi}{4}}$



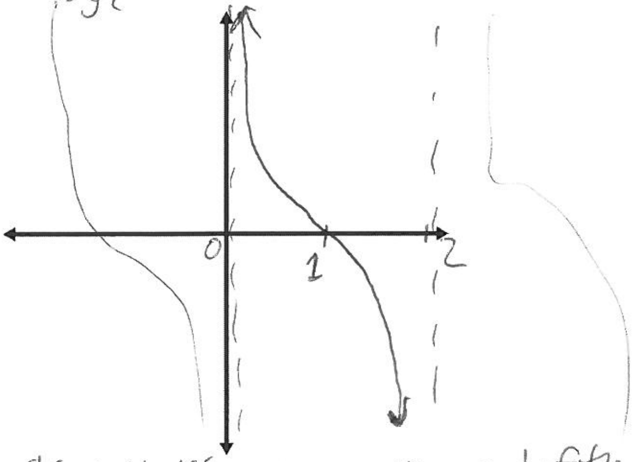
New X int: $0 + \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$

Period = $\frac{\pi}{k} = \frac{\pi}{1} = \pi$

$y = \cot(\frac{\pi}{2}x)$

Period: $\frac{\pi}{k} = \frac{\pi}{\frac{\pi}{2}} = \frac{\pi}{1} \cdot \frac{2}{\pi} = 2$

⇒ instead of graphing on $[0, \pi]$
graph on $[0, 2]$



asymptotes are on either end of the period.

X-intercept is in the middle

Since normal max/min at ±1

New max/min at $\pm a$

- Adjust asymp. based on period $\frac{2\pi}{k}$

Transformations of Cosecant and Secant

COSECANT AND SECANT CURVES

The functions

$$y = a \csc kx \qquad \text{and} \qquad y = a \sec kx \qquad (k > 0)$$

have period $2\pi/k$.

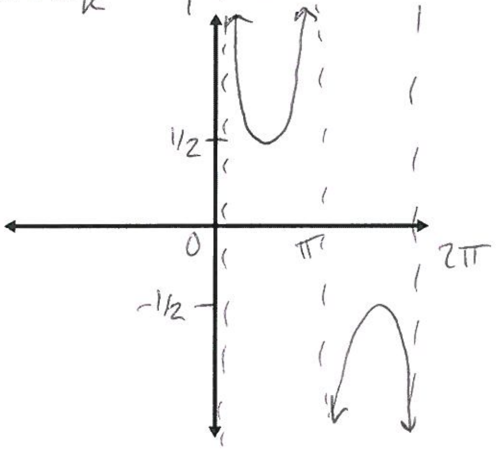
Using information from section 2.5 paired with the general forms seen on the first page we can graph these functions

$y = \frac{1}{2}\csc x$

$\frac{1}{2}$ flattens $\csc x$

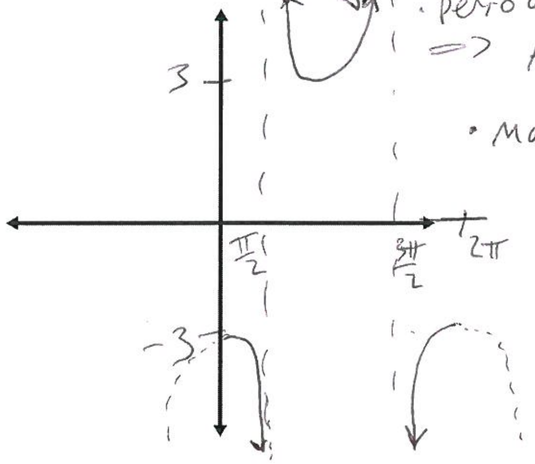
Period: $\frac{2\pi}{k} = \frac{2\pi}{1} = 2\pi$

Max/min at $\pm\frac{1}{2}$ $= \pm a$



$y = -3\sec x$

- −3 flips sec x upside down and stretches it vertically
- period is normal ⇒ Asym. normal
- max/min at $\pm a = \pm 3$

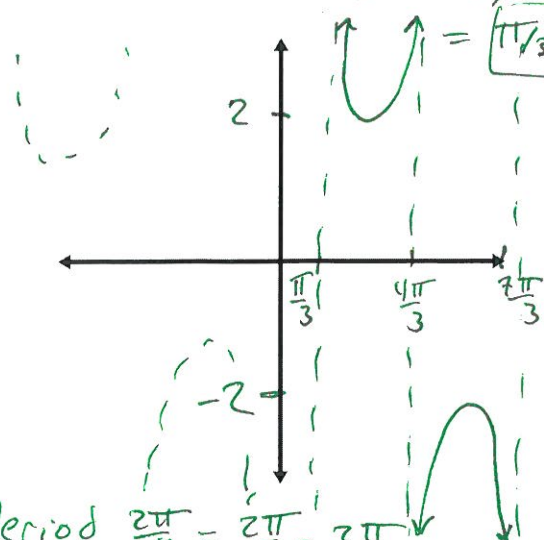


$y = 2\csc(x - \frac{\pi}{3})$

2 creates new max/min at ±2

$-\frac{\pi}{3}$ shift everything right $\frac{\pi}{3}$ units

New asymp. at $0+\frac{\pi}{3}$, $\pi+\frac{\pi}{3}$, $2\pi+\frac{\pi}{3}$ $= \pi/3, \frac{4\pi}{3}, \frac{7\pi}{3}$

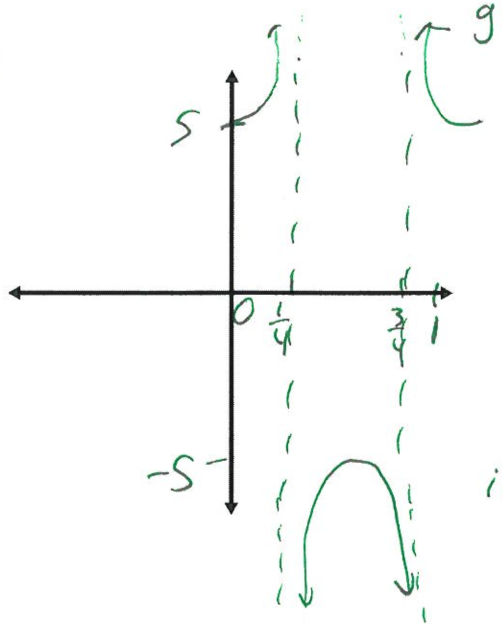


Period $\frac{2\pi}{k} = \frac{2\pi}{1} = 2\pi$

$y = 5\sec(2\pi x)$

New max/min ±5

Period: $\frac{2\pi}{k} = \frac{2\pi}{2\pi} = 1$ → instead of graphing on $[0, 2\pi]$, graph on $[0, 1]$



- asymptotes of sec occur $\frac{1}{4}$ distance from each end of the period ie at $\frac{1}{4}$, $\frac{3}{4}$ here

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5.5- Inverse Trig Functions and Their Graphs

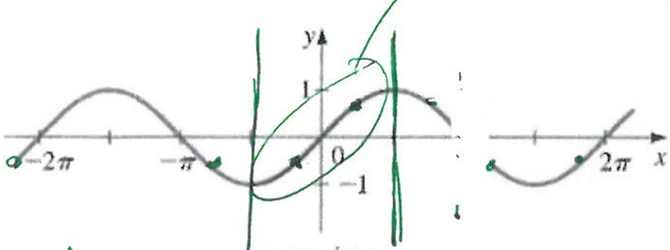
Recall, an inverse function of a graph sends all of its y-values back to the x-values from which they came.

That is, if $f(x) = y$, then $f^{-1}(y) = x$.

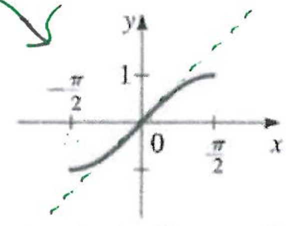
However, in order for this to work the function needs to be one-to-one.

Although Trig functions are not one-to-one, if we restrict their domain we can make them one-to-one on certain intervals, which will result in each having an inverse on a given interval.

Arcsine: The Inverse Sine Function



$y = \sin x$



$y = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

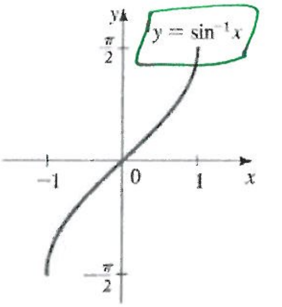
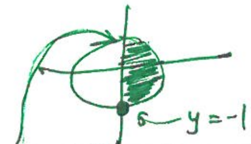


FIGURE 2 Graph of $y = \sin^{-1} x$



DEFINITION OF THE INVERSE SINE FUNCTION

The **inverse sine function** is the function $\sin^{-1}$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ defined by

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$

The inverse sine function is also called **arcsine**, denoted by **arcsin**.

$$\sin(\sin^{-1} x) = x \quad \text{for} \quad -1 \le x \le 1$$

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$\sin^{-1}(-1) = y$

Think "$\sin t = -1$ for what value of t?"

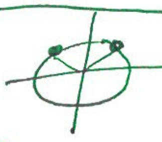
if $\sin t = -1$, we need y-value to $= -1$

This happens when $t = \frac{3\pi}{2}, -\frac{\pi}{2}, \frac{7}{2}\pi \ldots$ etc

Since we want $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\boxed{\sin^{-1}(-1) = -\frac{\pi}{2}}$

$\sin^{-1}(\frac{\sqrt{2}}{2})$

$\sin t = \frac{\sqrt{2}}{2}$ For what t?



We need $y = \frac{\sqrt{2}}{2}$, happens for $t = \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{5\pi}{4}$ ... etc

Choose $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

So choose $\frac{\pi}{4}$

$\boxed{\sin^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}}$

$\sin^{-1}(\sin(-\frac{\pi}{4}))$ is $-\frac{\pi}{4} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$?

yes so $\boxed{\sin^{-1}(\sin(-\frac{\pi}{4})) = -\frac{\pi}{4}}$

$\sin(\sin^{-1}(5))$ is $5 \in [-1, 1]$? NO

So first calculate $\sin^{-1}(5)$.

What value of t yields a y-value of 5? None, not possible.

Thus $\sin(\sin^{-1}(5))$ Does NOT exist

$\sin^{-1}(\sin(\frac{11\pi}{4}))$

is $\frac{11\pi}{4} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$? NO

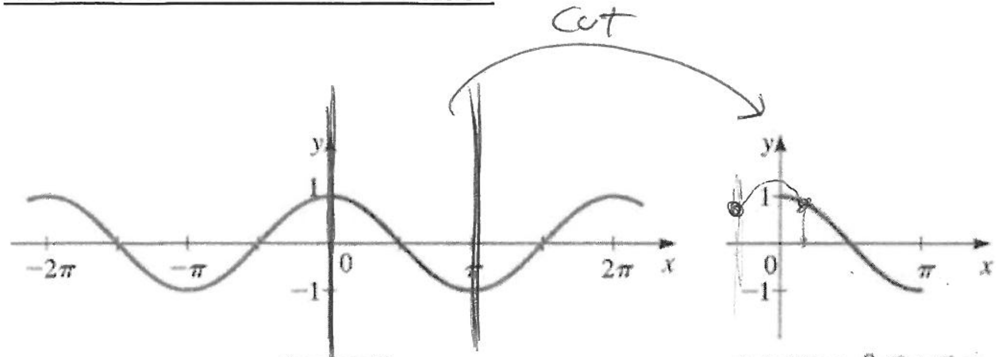
So calculate $\sin(\frac{11\pi}{4})$ 1st

$\frac{11\pi}{4}$ has t.p. $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ so

$\sin(\frac{11\pi}{4}) = y = \frac{\sqrt{2}}{2}$

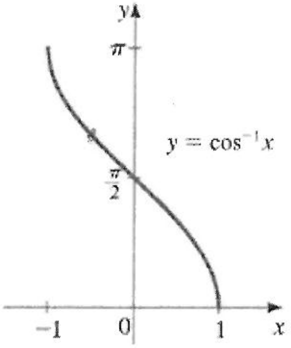
$\boxed{\sin^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}}$

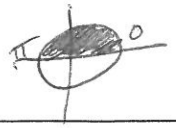
Arccosine: The Inverse Cosine Function



y = cos x

y = cos x, 0 ≤ x ≤ π





DEFINITION OF THE INVERSE COSINE FUNCTION

The **inverse cosine function** is the function $\cos^{-1}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

$$\cos^{-1}x = y \quad \Leftrightarrow \quad \cos y = x$$

The inverse cosine function is also called **arccosine**, denoted by **arccos**.

$\cos^{-1}(-1)$

$\cos t = -1$ for what value of t?

Need $x = -1$ on unit circle

↳ happens for $t = \pi, 3\pi, 5\pi, -\pi, \ldots$ etc

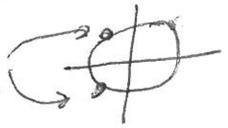
we need $t \in [0, \pi]$

$\Rightarrow \boxed{\cos^{-1}(-1) = \pi}$

$\cos^{-1}(-\frac{\sqrt{2}}{2})$

$\cos t = -\frac{\sqrt{2}}{2}$ for what t?

need $x = -\frac{\sqrt{2}}{2}$



$\Rightarrow t = \frac{3\pi}{4}, -\frac{3\pi}{4}$

.... etc

Since need $t \in [0, \pi]$ choose $\frac{3\pi}{4}$

$\boxed{\cos^{-1}(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}}$

note $\boxed{\cos^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}}$

$$\begin{array}{ll} \cos(\cos^{-1}x) = x \quad \text{for} & \boxed{-1 \le x \le 1} \\ \cos^{-1}(\cos x) = x \quad \text{for} & \boxed{0 \le x \le \pi} \end{array}$$

$\cos^{-1}(\cos(-\frac{\pi}{6}))$ is $-\frac{\pi}{6} \in [0, \pi]$?

NO, so calculate $\cos(-\frac{\pi}{6})$

$\cos(-\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$\boxed{\cos^{-1}(\frac{\sqrt{3}}{2}) = \pi/6}$

$\cos(\cos^{-1}(-\frac{2}{3}))$

is $-\frac{2}{3} \in [-1, 1]$? yes!

$\Rightarrow \cos(\cos^{-1}(-\frac{2}{3})) = -\frac{2}{3}$

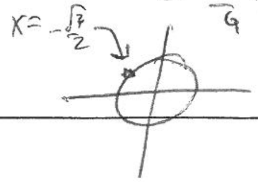
$\cos^{-1}(\cos(\frac{17\pi}{6}))$

$\frac{17\pi}{6} \in [0, \pi]$? NO

First find $\cos(\frac{17\pi}{6}) = -\frac{\sqrt{3}}{2}$

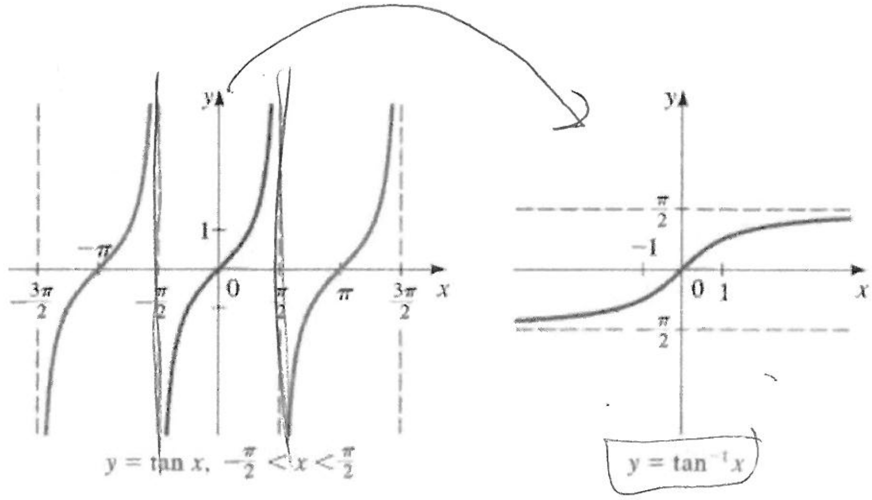
$\cos^{-1}(\cos(\frac{17\pi}{6})) = \cos^{-1}(-\frac{\sqrt{3}}{2})$

$= \frac{5}{6}\pi$



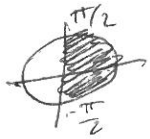
Arctangent: The Inverse Tangent Function

Cut as take inverse



$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

$y = \tan^{-1} x$



DEFINITION OF THE INVERSE TANGENT FUNCTION

The **inverse tangent function** is the function $\tan^{-1}$ with domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$ defined by

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

The inverse tangent function is also called **arctangent**, denoted by **arctan**.

$\tan^{-1}(-\frac{\sqrt{3}}{2})$ what t ~~results~~ in $\tan t = \frac{-\sqrt{3}}{2}$?

need t where $\frac{y}{x} = -\frac{\sqrt{3}}{2}$. The only $\sqrt{3}$ in numerator is if $y = \pm\frac{\sqrt{3}}{2}$. $y = \pm\frac{\sqrt{3}}{2}$ goes w/ $x = \pm\frac{1}{2}$

$(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ are associated w/ $t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{-\pi}{3}\ldots$ etc

since we want $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\boxed{\tan^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}}$

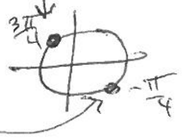
$\boxed{\tan^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}}$

$\tan^{-1}(-1)$

$\tan t = -1$ for what t? $\frac{y}{x} = -1 \Rightarrow \boxed{x = -y}$

$t = \frac{3\pi}{4}, -\frac{\pi}{4}, \frac{7\pi}{4} \ldots$ etc

need $t \in [-\pi/2, \pi/2] \Rightarrow \boxed{\tan^{-1}(-1) = -\frac{\pi}{4}}$



$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ → for practice

$\tan(\tan^{-1} x) = x \quad \boxed{\text{for } x \in \mathbb{R}}$

$\tan^{-1}(\tan x) = x \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$

$\tan^{-1}(\tan(-\frac{\pi}{4}))$ $-\frac{\pi}{4} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$?

yes so $= -\frac{\pi}{4}$

$\tan(\tan^{-1}(9846)) = 9846$

$\tan^{-1}(\tan(\frac{4\pi}{3})$

$\frac{4\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$? No

so first calc $\tan\frac{4\pi}{3}$
$= \sqrt{3}$

$\tan^{-1}(\sqrt{3})$

6.1- Angle Measure

θ measured in degrees = length of arc measured in radians

Angle Measure

Try to relate everything to unit circle

DEFINITION OF RADIAN MEASURE

If a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle in **radians** (abbreviated **rad**) is the length of the arc that subtends the angle (see Figure 2).

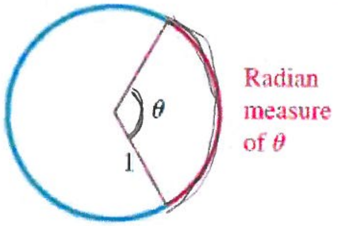
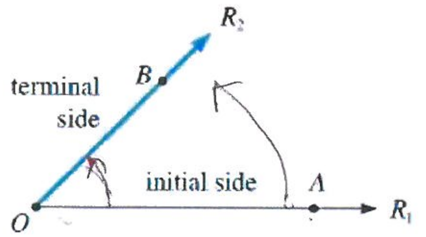
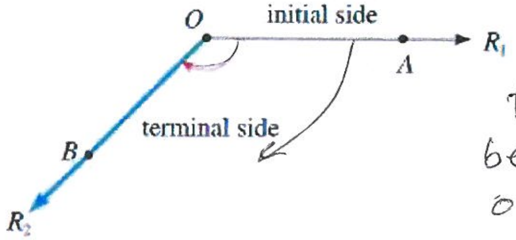


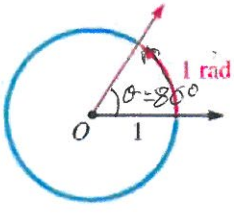
FIGURE 2 Usually we measure angles in degrees ex 90°. If we convert degrees to radians the Angle becomes equal to length of the arc

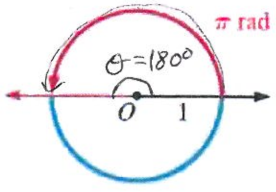


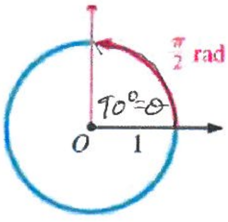
Positive angle

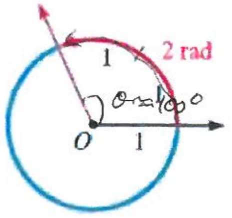


Negative angle









$\boxed{360° = 2\pi \text{ rad}}$

Since a complete revolution in degrees is 360° and in radians is 2π, the following relationship exists:

RELATIONSHIP BETWEEN DEGREES AND RADIANS

$$180° = \pi \text{ rad} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)° \qquad 1° = \frac{\pi}{180} \text{ rad}$$

1. To convert degrees to radians, multiply by $\frac{\pi}{180}$.
2. To convert radians to degrees, multiply by $\frac{180}{\pi}$.

Convert the angles given in degrees to angles in radians:

$54° \cdot \frac{\pi}{180} = \frac{6\pi}{20} = \boxed{\frac{3\pi}{10} \text{ rad}}$

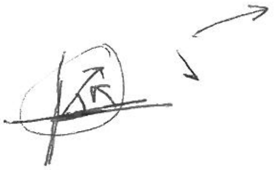
$-300° \cdot \frac{\pi}{180°} = \frac{-30\pi}{18} = \boxed{-\frac{5\pi}{3} \text{ rad}}$

Convert the angles given in radians to angles in degrees:

$\frac{11\pi}{3} \cdot \frac{180}{\pi} = \frac{11 \cdot 60}{1} = 660°$

$-\frac{3\pi}{2} \cdot \frac{180}{\pi} = -3 \cdot 90$
$= -270°$

Angles in Standard Position



An angle is in standard position if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis

Two angles in standard position are coterminal if their sides coincide

- Given an angle in radians one can determine other positive/negative angles that are coterminal by adding/subtracting multiples of 2π from the given angle
- Given an angle in degrees one can determine other positive/negative angles that are coterminal by adding/subtracting multiples of 360° from the given angle

Let $\theta = \frac{11\pi}{6}$, find some positive and negative coterminal angles

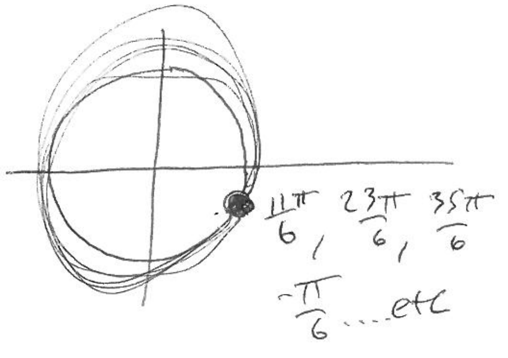
Note: $\boxed{2\pi = \frac{12\pi}{6}}$

$\frac{11\pi}{6} + 2\pi = \frac{11\pi}{6} + \frac{12\pi}{6} = \frac{23\pi}{6}$

$\frac{23\pi}{6} + \frac{12\pi}{6} = \boxed{\frac{35\pi}{6}} + \frac{12\pi}{6} = \boxed{\frac{47\pi}{6}}$

$\frac{11\pi}{6} - \frac{12\pi}{6} = \boxed{\frac{-\pi}{6}} - \frac{12\pi}{6} = \boxed{\frac{-13\pi}{6}}$

etc.....



Let $\theta = -45°$, find some positive and negative coterminal angles

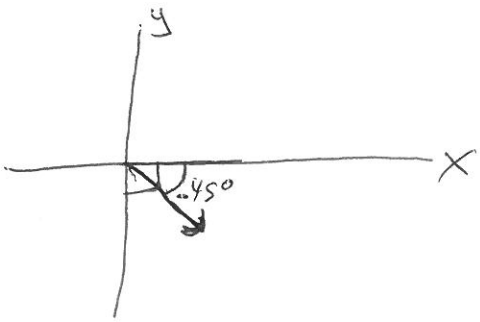
$-45° + 360° = \boxed{315°}$

$+360° = \boxed{675°}$

$675° + 360° = \boxed{1035°}$ etc...

$-45° - 360° = \boxed{-405°}$

$-405° - 360° = \boxed{-765°}$ etc...

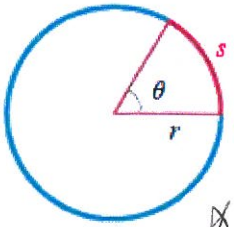


what is -45° in radians?
its half way From x-axis to y-axis in negative direction

$\Rightarrow \boxed{t = -\frac{\pi}{4}}$?

Check: $-45° \cdot \frac{\pi}{180°} = \frac{-1\pi}{4} = \frac{-\pi}{4}$ ✓

Length of a Circular Arc

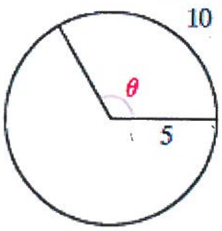
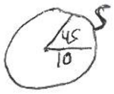


LENGTH OF A CIRCULAR ARC

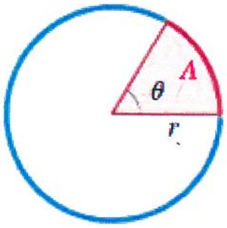
In a circle of radius r, the length s of an arc that subtends a central angle of θ radians is

$$s = r\theta$$

✱ $\boxed{\theta \text{ measured in radians}}$

Find the angle θ in the figure.  $S = r\theta$ $10 = 5\cdot\theta$ $\boxed{2 \text{ radians} = \theta}$ in degrees, $2\cdot\frac{180}{\pi} = \boxed{\frac{360^\circ}{\pi} = \theta}$	Find the length of an arc that subtends a central angle of 45°in a circle of radius 10 m  need $\theta = 45^\circ$ into radians $45\cdot\frac{\pi}{180} = \boxed{\frac{\pi}{4}}$ rad $S = \theta r = \frac{\pi}{4}\cdot 10$ $\boxed{S = \frac{5\pi}{2} \text{ rad}}$	A central angle θ in a circle of radius 5 m is subtended by an arc of length 6 m. Find the measure of θ in degrees and radians $S = r\theta$ $6 = 5\cdot\theta \Rightarrow \boxed{\theta = \frac{6}{5} \text{ rad}}$ $\frac{6}{5}\cdot\frac{180}{\pi} = \boxed{\frac{216^\circ}{\pi} \approx 70^\circ}$

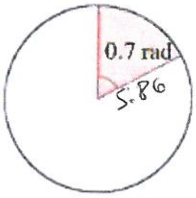
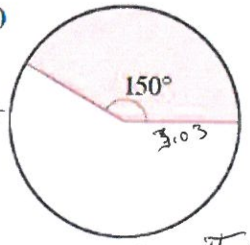
Area of a Circular Sector



AREA OF A CIRCULAR SECTOR

In a circle of radius r, the area A of a sector with a central angle of $\boxed{\theta \text{ radians}}$ is

$$A = \frac{1}{2}r^2\theta$$

Find the radius of each circle if the area of the sector is 12. **(a)**  $A = \frac{1}{2}r^2\theta$ **(b)**  $12 = \frac{1}{2}r^2(.7)$ $r = \sqrt{\frac{24}{.7}} \approx 5.86$ $150\cdot\frac{\pi}{180} \approx 2.61 \text{ rad}$ $12 = \frac{1}{2}r^2(2.61)$ $r = \sqrt{\frac{24}{2.61}} \approx 3.03$	If a circle has a sector of area 10 and radius 2 calculate the central angle θ in radians and degrees $10 = \frac{1}{2}(2)^2\theta$ $10 = 2\theta$ $\boxed{\theta = 5 \text{ rad}}$ $5\cdot\frac{180}{\pi} = \boxed{286.5^\circ = \theta}$

$\sqrt{\frac{24}{2.61}}$

6.2- Trigonometry of Right Triangles

Trigonometric Ratios

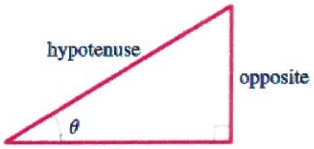


FIGURE 1

THE TRIGONOMETRIC RATIOS

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

① hypotenuse always across right angle
② adjacent - touches the angle
③ opposite - not ① or ② its opposite θ

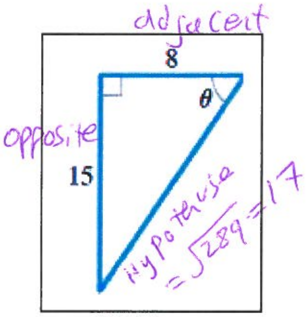
S O H — sin, opposite, hypot.
C A H — cos, adjacent, hypot
T O A — tangent, opposite, adjacent

cHo SHa cao

• Since $\csc = \frac{1}{y} = \frac{1}{\sin}$

$\Rightarrow \csc = \frac{1}{\frac{opp}{Hyp}} = \frac{Hyp}{opp}$

• $\sec = \frac{1}{\cos} = \frac{1}{\frac{adj}{Hyp}} = \frac{Hyp}{adj}$



Find the exact values of the six trigonometric ratios of the angle θ in the triangle

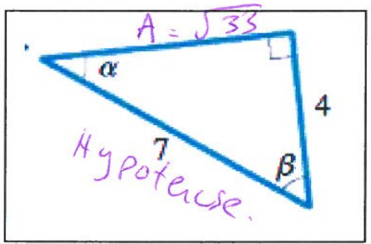
First Find the hypothenuse using Pythagorean Theorem

$15^2 + 8^2 = H^2 \Rightarrow \boxed{\sqrt{287} = H} = 17$

① $\sin\theta = \frac{opp}{Hyp} = \frac{15}{17}$ ② $\cos\theta = \frac{8}{17}$ ③ $\tan\theta = \frac{15}{8}$

④ $\csc\theta = \frac{17}{15}$ ⑤ $\sec\theta = \frac{17}{8}$ ⑥ $\cot\theta = \frac{8}{15}$

Consider the following triangle:



Find side A: $A^2 + 4^2 = 7^2$

$\boxed{A = \sqrt{33}}$

b) Find sin α and cos β	a) Find tan α and cot β	c) Find sec α and csc β
$\sin\alpha = \frac{opp}{Hyp} = \frac{4}{7}$ $\cos\beta = \frac{adj}{Hyp} = \frac{4}{7}$	$\tan\alpha = \frac{opp}{Hyp} = \frac{4}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = \frac{4\sqrt{33}}{33}$ $\cot\beta = \frac{adj}{opp} = \frac{4}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = \frac{4\sqrt{33}}{33}$	$\sec\alpha = \frac{hyp}{adj} = \frac{7}{\sqrt{33}} = \frac{7\sqrt{33}}{33}$ $\csc\beta = \frac{hyp}{opp} = \frac{7}{\sqrt{33}} = \frac{7\sqrt{33}}{33}$

$\sin\beta = \frac{opp}{Hyp} = \frac{\sqrt{33}}{7}$

$\cos\alpha = \frac{adj}{Hyp} = \frac{\sqrt{33}}{7}$

good for quizzes/tests

For webassign

$\boxed{\text{Sum of all angles in a triangle} = 180^\circ}$

Special Triangles

You should remember these special angle ratios because they are the ones that we will use:

θ in degrees	θ in radians	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

I recommend remembering that $\frac{\pi}{6} = 30^\circ$, $\frac{\pi}{4} = 45^\circ$ and $\frac{\pi}{3} = 60^\circ$ and relate them to the unit circle

$\boxed{ex}$ $\sin(60^\circ) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

Application of Trigonometry of Right Triangles

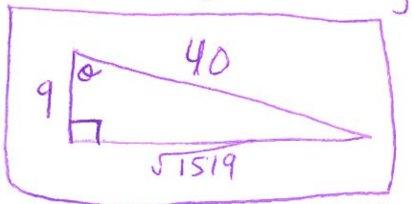
Given a trigonometric ratio we can construct the triangle

Ex: If $\cos\theta = \frac{9}{40}$, sketch a triangle that has acute angle θ

$\cos\theta = \frac{9}{40} = \frac{adj}{Hyp} \Rightarrow$ adj = 9, Hyp = 40 } → Solve for other side

$9^2 + B^2 = 40^2$

$B = \sqrt{1519} \approx 39$

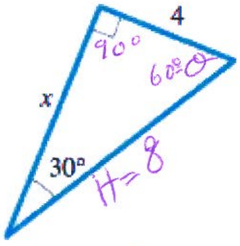


If a question asks you to "<u>solve the right triangle</u>" it means <u>find all side lengths and all angles</u>.

We can do this using all of the above properties paired with the Pythagorean Theorem

Ex: Solve the following right triangle

Sum $30^\circ + 90^\circ + \theta = 180^\circ \Rightarrow \boxed{\theta = 60^\circ}$



we need 2 sides to use Pyth. Theorem to find 3rd.

We know various values of trig functions for θ = 60°, 30° from table above.

different choices here, I'll sin(30) since 4 opposite 30 and sin = $\frac{opp}{hyp}$.

$\sin(30) = \frac{opp}{Hyp} = \frac{4}{H}$.

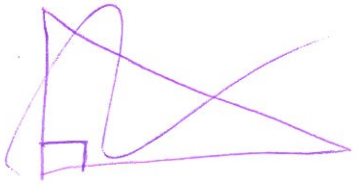
we know $\sin(30) = \sin(\pi/6) = \frac{1}{2}$

$\frac{1}{2} = \frac{4}{H} \Rightarrow \boxed{H = 8}$ → use pytha Thm to solve for x

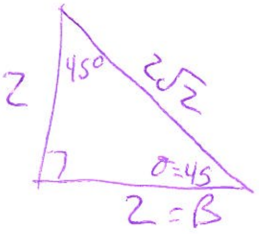
$\tan(30) = \tan(\frac{\pi}{6}) = \frac{\sqrt{3}}{3}$ (From chs)

$\tan(30) = \frac{opp}{adj} = \frac{4}{x}$

$\frac{\sqrt{3}}{3} = \frac{4}{x} \rightarrow x\sqrt{3} = 12 \Rightarrow \boxed{x = \frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{3}}$



Solve the right triangle



① Find missing angles

② use trig values in table to find a second side

③ Find 3rd side

① $45 + 90 + \theta = 180$

$\Rightarrow \theta = 180 - 90 - 45 = 45°$

② $\sin\theta = \frac{opp}{Hyp}$

$\sin(45) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$

$\sin(45) = \frac{2}{H}$ so $\frac{\sqrt{2}}{2} = \frac{2}{H}$

$\sqrt{2}H = 4 \Rightarrow H = \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = \boxed{2\sqrt{2} = H}$

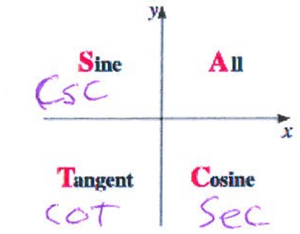
③ $A^2 + B^2 = C^2$ so $2^2 + B^2 = (2\sqrt{2})^2$

$4 + B^2 = \underbrace{4 \cdot 2}_{=8}$

$B^2 = 4 \Rightarrow \boxed{B=2}$

6.3- Trigonometric Functions of Angles

A few reminders:



You can remember this as "All Students Take Calculus."

SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

↳ use this to determine sign (+ or −) of a trig function in a given Quadrant

REFERENCE ANGLE

Let θ be an angle in standard position. The **reference angle** $\bar{\theta}$ associated with θ is the acute angle formed by the terminal side of θ and the x-axis.

→ Same idea as a reference number in a unit circle

To convert degrees to radians, multiply by $\frac{\pi}{180}$ → use this to evaluate trig functions at degree values

We can use the above techniques to determine the value of any trig function given an angle θ

EVALUATING TRIGONOMETRIC FUNCTIONS FOR ANY ANGLE

To find the values of the trigonometric functions for any angle θ, we carry out the following steps.

1. Find the reference angle $\bar{\theta}$ associated with the angle θ.
2. Determine the sign of the trigonometric function of θ by noting the quadrant in which θ lies.
3. The value of the trigonometric function of θ is the same, except possibly for sign, as the value of the trigonometric function of $\bar{\theta}$.

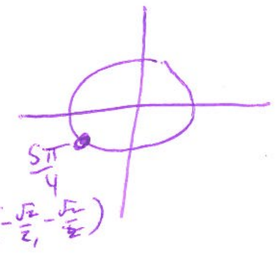
Ex-Find the exact values of the following functions:

sin 225°

convert 225° to radians

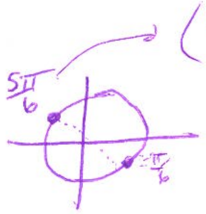
$$225° \cdot \frac{\pi}{180°} = \frac{45\pi}{36} = \frac{5\pi}{4}$$

$$\sin(225°) = \sin\left(\frac{5\pi}{4}\right)$$



$$= y = \boxed{-\frac{\sqrt{2}}{2}}$$

$\tan\frac{5\pi}{6}$



$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\cdot -\frac{2}{\sqrt{3}} = \boxed{-\frac{1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}}$$

FUNDAMENTAL IDENTITIES

remember, Many of these can be derived From the table on Pg 1 of 5.2

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

all forms of $x^2+y^2=1$ They re-arranged to form other identities

ex

$$\boxed{\begin{gathered}\sin^2\theta + \cos^2\theta = 1\\ \sin\theta = \pm\sqrt{1-\cos^2\theta}\\ \cos\theta = \pm\sqrt{1-\sin^2\theta}\end{gathered}}$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad \tan\theta = \pm\sqrt{\sec^2\theta - 1} \qquad \sec\theta = \pm\sqrt{\tan^2\theta+1}$$

$$1 + \cot^2\theta = \csc^2\theta \qquad \cot\theta = \pm\sqrt{\csc^2\theta-1} \qquad \csc\theta = \pm\sqrt{1+\cot^2\theta}$$

Using these fundamental identities, we can make substitutions to write one trig function in terms of another.

Write cot θ in terms of sin θ given that θ is in Quadrant II → Want $\cot\theta = \left(\text{some function of only } \sin\theta\right)$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ → need to get rid of this and change it to something w/ a $\sin\theta$ in it

from above, $\cos\theta = \pm\sqrt{1-\sin^2\theta}$

In Quad II $\cos < 0$

So $\cos\theta = -\sqrt{1-\sin^2\theta}$ in Quad II

$$\boxed{\cot\theta = \frac{-\sqrt{1-\sin^2\theta}}{\sin\theta}}$$

Alternative approach

$\cot\theta = \dfrac{1}{\tan\theta}$, we know $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\cot\theta = \dfrac{1}{\frac{\sin\theta}{\cos\theta}} = \dfrac{\cos\theta}{\sin\theta}$ → what we started w/ over here

Write sec θ in terms of sin θ given that θ is in Quadrant I

$\sec\theta = \dfrac{1}{\cos\theta}$ We know $\cos\theta = \pm\sqrt{1-\sin^2\theta}$

$\cos\theta > 0$ in Quad I So use $+\sqrt{1-\sin^2\theta}$

$\sec\theta = \dfrac{1}{\sqrt{1-\sin^2\theta}}$ in Quad I

$\sec\theta = \sqrt{\tan^2\theta+1}$

$\tan^2\theta = \left(\dfrac{\sin\theta}{\cos\theta}\right)^2 = \dfrac{\sin^2\theta}{\cos^2\theta}$

$\sec\theta = \sqrt{\dfrac{\sin^2\theta}{\cos^2\theta}+1}$

$\sec\theta = \sqrt{\dfrac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}} = \sqrt{\dfrac{1}{\cos^2\theta}}$

$\sec\theta = \dfrac{\sqrt{1}}{\sqrt{\cos^2\theta}} = \dfrac{1}{\cos\theta}$ → what we started w/ before

~~Write~~ Secθ

cotθ

Write ~~sec~~cotθ in terms of secθ in Quad III.

$\cot\theta = \dfrac{1}{\tan\theta}$ where $\tan\theta = \pm\sqrt{\sec^2\theta - 1}$

in Quad III $\cot\theta, \tan\theta > 0$

$\cot\theta = \dfrac{1}{\sqrt{\sec^2\theta - 1}}$ in Quad III

7.1- Trigonometric Identities

Many of the following identities we studied in previous sections:

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

Same x-values opposite y-values

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

From 5.2

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \sec\left(\frac{\pi}{2} - u\right) = \csc u$$

$$\cos\left(\frac{\pi}{2} - u\right) = \sin u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

(almost) never used

In the following examples, rewrite the expression in terms of sine and cosine and then simplify the expression.

Work	Substitution
$\tan(\theta)\csc(\theta)$ $= \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cancel{\sin\theta}} = \frac{1}{\cos\theta} = \boxed{\sec\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$
$\frac{\sec(x) - \cos(x)}{\tan(x)}$ $\frac{\frac{1}{\cos x} - \cos x}{\frac{\sin x}{\cos x}}\cdot\frac{\cos x}{\cos x} = \frac{\frac{\cos x}{\cos x} - \cos x^2}{\frac{\sin x\cos x}{\cos x}}$	$\sec\theta = \frac{1}{\cos\theta}$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$ since $\sin^2 x + \cos^2 x = 1$ $\Rightarrow \sin^2 x = 1 - \cos^2 x$

$$= \frac{1 - \cos^2 x}{\sin x} = \frac{\sin^2 x}{\sin x} = \sin x$$

Proving Trigonometric Identities

Many identities arrise from the fundamental identities shown on the first page. The idea behind proving an identity is that we are given two things that are said to be equal and we want to start with one side of the identity and transform it into the other side of the identity using substitutions from page 1.

GUIDELINES FOR PROVING TRIGONOMETRIC IDENTITIES

1. **Start with one side.** Pick one side of the equation and write it down. Your goal is to transform it into the other side. It's usually easier to start with the more complicated side.
2. **Use known identities.** Use algebra and the identities you know to change the side you started with. Bring fractional expressions to a common denominator, factor, and use the fundamental identities to simplify expressions.
3. **Convert to sines and cosines.** If you are stuck, you may find it helpful to rewrite all functions in terms of sines and cosines.

④ If you get stuck along the way try working backwards. That is, try starting from the less complicated side and work back towards the more complicated side.

Work	Reason
Show $\cos(-x) - \sin(-x) = \cos(x) + \sin(x)$ more complicated $\cos(-x) - \sin(-x)$ $= \cos(x) - (-\sin(x))$ $= \cos(x) + \sin(x)$	$\cos(-x) = \cos(x)$ $\sin(-x) = -\sin(x)$

Work	Reason
Show $\frac{\cos(x)}{\sec(x)} + \frac{\sin(x)}{\csc(x)} = 1$ $\frac{\cos(x)}{\sec(x)} + \frac{\sin(x)}{\csc(x)}$ $= \frac{\cos(x)}{\frac{1}{\cos(x)}} + \frac{\sin(x)}{\frac{1}{\sin(x)}}$ $= \cos x \cdot \cos x + \sin x \cdot \sin x$ →	$\sec x = \frac{1}{\cos x}$ $\csc x = \frac{1}{\sin x}$ Fraction Property

$= \cos^2 x + \sin^2 x$ → $\sin^2 x + \cos^2 x = 1$

$= 1$

Work	Reason
Show $\frac{1+\tan^2 x}{1-\tan^2 x} = \frac{1}{\cos^2 x-\sin^2 x}$	
$\frac{1+\tan^2 x}{1-\tan^2 x} = \frac{1+\frac{\sin^2 x}{\cos^2 x}}{1-\frac{\sin^2 x}{\cos^2 x}} \cdot \frac{\cos^2 x}{\cos^2 x}$	$\tan^2 x = \left(\frac{\sin x}{\cos x}\right)^2 = \frac{\sin^2 x}{\cos^2 x}$
$\frac{\cos^2 x+\sin^2 x}{\cos^2 x-\sin^2 x} = \frac{1}{\cos^2 x-\sin^2 x}$	$\sin^2 x+\cos^2 x = 1$

	Reason
$\frac{1+\tan^2 x}{1-\tan^2 x} = \frac{\sec^2 x}{1-\tan^2 x}$	$1+\tan^2 x = \sec^2 x$
$= \frac{\sec^2 x}{1-\frac{\sin^2 x}{\cos^2 x}}$	$\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$
$= \frac{\sec^2 x}{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}} = \sec^2 x \cdot \frac{\cos^2 x}{\cos^2 x-\sin^2 x}$	$\sec^2 x = \frac{1}{\cos^2 x}$
$= \frac{1}{\cancel{\cos^2 x}} \cdot \frac{\cancel{\cos^2 x}}{\cos^2 x-\sin^2 x} = \frac{1}{\cos^2 x-\sin^2 x}$	

Work	Reason
Show $(\cot x - \csc x)(\cos x + 1) = -\sin x$	
$((\cot x - \csc x)(\cos x + 1)$	distribute
$\cot x \cos x + \cot x - \csc x \cos x - \csc x$	$\cot x = \frac{\cos x}{\sin x}$ $\csc x = \frac{1}{\sin x}$
$\frac{\cos x}{\sin x} \cdot \cos x + \cancel{\frac{\cos x}{\sin x}} - \cancel{\frac{\cos x}{\sin x}} - \frac{1}{\sin x}$	
$= \frac{\cos^2 x}{\sin x} - \frac{1}{\sin x}$	$\cos^2 x + \sin^2 x = 1$ $\boxed{\cos^2 x - 1 = -\sin^2 x}$
$= \frac{\cos^2 x - 1}{\sin x} = \frac{-\sin^2 x}{\sin x}$	
$= -\sin x$	

Conjugate of $a+bx$ is $a-bx$
$a-bx$ is $a+bx$

Notice the following "hidden identities":

- If given $(1-\sin x)$, notice that $(1-\sin x)(1+\sin x) = 1-\sin^2 x$ and since $\sin^2 x + \cos^2 x = 1$, $1-\sin^2 x = \cos^2 x$ and thus $\boxed{(1-\sin x)(1+\sin x) = 1-\sin^2 x = \cos^2 x}$
- Similarly, if given $(1-\cos x)$ then $\boxed{(1-\cos x)(1+\cos x) = 1-\cos^2 x = \sin^2 x}$
- Also, if given $(\sec x - 1)$, then $\boxed{(\sec x - 1)(\sec x + 1) = \sec^2 x - 1 = \tan^2 x}$ (since $\tan^2 x + 1 = \sec^2 x$)
- Similarly, if given $\csc x - 1$ then $\boxed{(\csc x - 1)(\csc x + 1) = (\csc^2 x - 1) = \cot^2 x}$

Moral: You can re-arrange the identities on the first page to create new identities.

☆ Also: given $[1 \pm (\cos x / \sin x / \sec x / \csc x / others)]$ you may have to multiply by the conjugate to move forward in proving the identity. ☆

Work	Reason
Show $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2\sec x \tan x$	
$\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = \frac{1}{1-\sin x} \cdot \frac{(1+\sin x)}{(1+\sin x)} - \frac{1}{1+\sin x} \frac{(1-\sin x)}{(1-\sin x)}$	find common denominator $\Rightarrow (1-\sin x)(1+\sin x)$
$= \frac{(1+\sin x)}{\cos^2 x} - \frac{(1-\sin x)}{\cos^2 x}$	from above, $(1+\sin x)(1-\sin x) = \cos^2 x$
$= \frac{1+\sin x - 1 + \sin x}{\cos^2 x} = \frac{2\sin x}{\cos^2 x}$	added fractions
If stuck, try working backwards $= 2\,\frac{\sin x}{\cos^2 x}$	
$= 2\left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)$	
$2\sec x \tan x$	$\sec x = \frac{1}{\cos x}$ $\tan x = \frac{\sin x}{\cos x}$

Work	Reason
Show $\frac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$	Since $1-\sin\theta$ in denominator, multiply by $1+\sin\theta$ ad use Property From prev. Page
① $\frac{\cos\theta}{1-\sin\theta}\cdot\frac{(1+\sin\theta)}{(1+\sin\theta)} = \frac{\cos\theta+\cos\theta\sin\theta}{\cos^2\theta}$	
$= \frac{\cos\theta}{\cos^2\theta} + \frac{\cos\theta\sin\theta}{\cos^2\theta}$	fraction property
$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$	$\sec\theta = \frac{1}{\cos\theta}$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$
② $\sec\theta + \tan\theta$ $\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} =$	$\sec\theta = \frac{1}{\cos\theta}$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$
$= \frac{1+\sin\theta}{\cos\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right) = \frac{\cos^2\theta}{\cos\theta - \cos\theta\sin\theta}$	from prev. Pg
$= \frac{\cos^2\theta}{\cos\theta(1-\sin\theta)} = \frac{\cos\theta}{1-\sin\theta}$	cancelling $\cos\theta$

Make the indicated substitution and simplify.

$\sqrt{1+x^2}$, $x = \tan\theta$. Assume that $0 \le \theta < \frac{\pi}{2}$

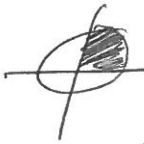
If $x = \tan\theta \Rightarrow \pm\sqrt{1+x^2} = \pm\sqrt{1+\tan^2\theta}$ where $1+\tan^2\theta = \sec^2\theta$

$= \pm\sqrt{\sec^2\theta}$

$= \pm\sec\theta$

$\boxed{= \sec\theta}$ ←

$\theta \in [0, \frac{\pi}{2}] \rightarrow$



everything positive in Quad I

$\frac{1}{x^2\sqrt{4+x^2}}$, $x = 2\tan\theta$. Assume that $0 \le \theta < \frac{\pi}{2}$. → againg all trig functions positive here

$$\frac{1}{(2\tan\theta)^2\sqrt{4+(2\tan\theta)^2}} = \frac{1}{4\tan^2\theta\sqrt{4+4\tan^2\theta}} = \frac{1}{4\tan^2\theta\sqrt{4(1+\tan^2\theta)}}$$

factor

$$\frac{1}{4\tan^2\theta\sqrt{4\sec^2\theta}} = \frac{1}{4\tan^2\theta\, 2\sec\theta} = \frac{1}{8\left(\frac{\sin^2\theta}{\cos^2\theta}\right)\left(\frac{1}{\cos\theta}\right)}$$

$1+\tan^2\theta = \sec^2\theta$

take square root

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

$$= \frac{1}{8\left(\frac{\sin^2\theta}{\cos^3\theta}\right)} = \boxed{\frac{\cos^3\theta}{8\sin^2\theta}}$$

combined fractions

fraction property

7.2- Addition and Subtraction Formulas

The following formulas do not need to be memorized as they will be provided on quizzes/tests/exams

recall, $60° = \frac{\pi}{3}$, $45° = \frac{\pi}{4}$

$30° = \frac{\pi}{6}$

ADDITION AND SUBTRACTION FORMULAS

Formulas for sine:

① $\sin(s + t) = \sin s \cos t + \cos s \sin t$

② $\sin(s - t) = \sin s \cos t - \cos s \sin t$

Formulas for cosine:

③ $\cos(s + t) = \cos s \cos t - \sin s \sin t$

④ $\cos(s - t) = \cos s \cos t + \sin s \sin t$

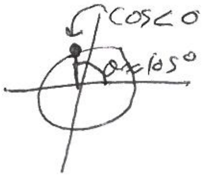
Formulas for tangent:

⑤ $\tan(s + t) = \dfrac{\tan s + \tan t}{1 - \tan s \tan t}$

⑥ $\tan(s - t) = \dfrac{\tan s - \tan t}{1 + \tan s \tan t}$

If asked to evaluate a sin, cos or tan at a degree other than 30, 60, 45 then use formulas on left to transform it into something w/ 30, 60, 45

We can use the above identities to evaluate an unknown trig ~~identity~~ value by transforming them to a more familiar form :



Evaluate cos(105°)

$105° = 60° + 45°$ so $\cos(105°) = \cos(60° + 45°)$

use (#3)

$\cos(105) = \cos(\underbrace{60}_{s} + \underbrace{45}_{t}) = \cos(60).\cos(45) - \sin(60)\sin(45)$ by ③

remember, $60° = \frac{\pi}{3}$ rad → $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ $45° = \frac{\pi}{4}$ rad → $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$

$\cos(60) = \frac{1}{2}$ $\cos(45) = \frac{\sqrt{2}}{2}$

$\sin(60) = \frac{\sqrt{3}}{2}$ $\sin(45) = \frac{\sqrt{2}}{2}$

so $\cos(105) = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \boxed{\frac{\sqrt{2} - \sqrt{6}}{4}}$

Evaluate cos(10°) cos(80°) − sin(10°) sin(80°)

By③ $\cos(10)\cos(80) - \sin(10)\sin(80) = \cos(10 + 80) = \cos(90)$

$90° = \frac{\pi}{2}$ rad $\Rightarrow (0, 1)$

$\cos(90) = 0$

original = 0

$\sin(15°) = \sin(60 - 45)$

or $= \sin(45 - 30)$

By ② $\sin(60 - 45) = \sin 60 \cdot \cos 45 - \cos 60 \cdot \sin 45$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

We can use the addition and subtraction formulas to prove identities.

Prove that $\cot\left(\frac{\pi}{2} - u\right) = \tan u$

use ④ ↗

$$\cot x = \frac{\cos x}{\sin x} \quad \text{So} \quad \cot\left(\frac{\pi}{2} - u\right) = \frac{\cos(\pi/2 - u)}{\sin(\pi/2 - u)} = \frac{\cos\frac{\pi}{2}\cos u + \sin\frac{\pi}{2}\sin u}{\sin\frac{\pi}{2}\cos u - \cos\frac{\pi}{2}\sin u}$$

↓ use ②

$$\boxed{\frac{\pi}{2} \Rightarrow (0,1)}$$

$$= \frac{0 \cdot \cos u + 1 \cdot \sin u}{1 \cdot \cos u - 0 \cdot \sin u} = \frac{\sin u}{\cos u} = \tan u$$

Prove that $\underbrace{\cos(x + y)}_{\text{use ③}} + \underbrace{\cos(x - y)}_{\text{use ④}} = 2\cos x \cos y$

$$\cos(x+y) + \cos(x-y) = \overset{③}{[\cos(x)\cos(y) - \cancel{\sin x \sin y}]} + \overset{④}{[\cos x \cos y + \cancel{\sin x \sin y}]}$$

$$= 2\cos x \cos y$$

Prove that $\dfrac{\overset{①}{\overbrace{\sin(x+y)}} - \overset{②}{\overbrace{\sin(x-y)}}}{\underset{③}{\underbrace{\cos(x+y)}} + \underset{④}{\underbrace{\cos(x-y)}}} = \tan y$

use

use

cancel things

$$\frac{\sin(x+y) - \sin(x-y)}{\cos(x+y) + \cos(x-y)}$$

plug in ①,②,③,④ from pg 1

$$\frac{\overset{①}{[\cancel{\sin x \cos y} + \cos x \sin y]} - \overset{②}{[\cancel{\sin x \cos y} - \cos x \sin y]}}{\underset{③}{[\cos x \cos y - \cancel{\sin x \sin y}]} + \underset{④}{[\cos x \cos y + \cancel{\sin x \sin y}]}}$$

$$= \frac{\cancel{2}\cancel{\cos x}\sin y}{\cancel{2}\cancel{\cos x}\cos y} = \frac{\sin y}{\cos y} = \tan y$$

7.3- Double-Angle, Half-Angle and Product-Sum Formulas

You **will** need to memorize the following formulas as they will **not** be provided on tests/exams

DOUBLE-ANGLE FORMULAS $\sin(2\text{thing}) = 2\sin(\text{thing})\cos(\text{thing})$

Formula for sine: $\sin 2x = 2\sin x\cos x$

Formulas for cosine: $\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

Formula for tangent: $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Similar to 7.2, these formulas help us evaluate trig functions at difficult values and help prove identities

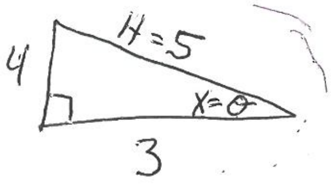
We can use the Double-Angle Formulas to solve problems like this: In problems using double angle formulas, draw the right triangle first

Ex:

Find $\sin(2x)$, $\cos(2x)$, and $\tan(2x)$ from the given information:

$\tan(x) = -\frac{4}{3}$ and x is in Quadrant II

$\tan x = \frac{opp}{adj} = \frac{-4}{3} \longrightarrow$



Find 3rd side: $4^2 + 3^2 = H^2$

$\Rightarrow H = \sqrt{25} = 5$

$\cos x = \frac{adj}{Hyp} = \frac{3}{5}$ in Quad II $= -\frac{3}{5}$

$\sin x = \frac{4}{5}$ in Quad II

$$\sin(2x) = 2\sin x\cos x = 2\left(\tfrac{4}{5}\right)\left(-\tfrac{3}{5}\right) = \boxed{\frac{-24}{25}}$$

$$\cos(2x) = \cos^2 x - \sin^2 x = \left(\tfrac{3}{5}\right)^2 - \left(\tfrac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = \boxed{-\frac{7}{25}}$$

$$= 1 - 2\sin^2 x = 1 - 2\left(\tfrac{16}{25}\right) = \frac{25}{25} - \frac{32}{25} = \boxed{-\frac{7}{25}}$$

$$= 2\cos^2 x - 1 = 2\left(-\tfrac{3}{5}\right)^2 - 1 = \frac{18}{25} - \frac{25}{25} = \boxed{-\frac{7}{25}}$$

$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\left(-\frac{4}{3}\right)}{1 - \left(-\frac{4}{3}\right)^2} = \frac{-\frac{8}{3}}{\frac{9}{9} - \frac{16}{9}} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = \boxed{\frac{24}{7}}$$

FORMULAS FOR LOWERING POWERS

Not to Memorize

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

We can use the formulas for lowering powers to rewrite expression that involves high powers to those involving the first power of cosine.

Ex:

Use the formulas for lowering powers to rewrite the expression in terms of the first power of cosine.

$$\cos^4 x = (\cos^2(x))^2 = \left(\frac{1+\cos 2x}{2}\right)^2 = \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1+2\cos 2x+\cos^2(2x)}{4} = \frac{1+2\cos(2x)+\boxed{\frac{1+\cos(4x)}{2}}}{4} \longrightarrow$$

From above

$$\cos^2(2x) = \frac{1+\cos(2(2x))}{2}$$

Mult by $\frac{2}{2}$

$$= \frac{2+2\cdot 2\cos(2x)+(1+\cos 4x)}{4\cdot 2} = \frac{3+4\cos 2x+\cos 4x}{8}$$

HALF-ANGLE FORMULAS

Not to Memorize

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The choice of the + or − sign depends on the quadrant in which $u/2$ lies.

Another approach to evaluating trig functions of strange values

Use an appropriate Half-Angle formula to find the exact value of the expressions:

(tan(45−30))

$$\tan 15° = \tan\left(\frac{30}{2}\right) \Rightarrow u = 30$$

$$= \frac{1-\cos(30)}{\sin(30)}$$

$$= \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \cdot \frac{2}{2} = \frac{2-\sqrt{3}}{1}$$

$$= \boxed{2-\sqrt{3}}$$

Quad I

u=45

$$\sin 22.5° = \sin\left(\frac{45}{2}\right)$$

$$\sin\left(\frac{45}{2}\right) = \pm\sqrt{\frac{1-\cos 45}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$= \pm\sqrt{\frac{2-\sqrt{2}}{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{2}$$

22.5 in Quad I, sin > 0

$$\text{so} = +\frac{\sqrt{2-\sqrt{2}}}{2}$$

Prove the following Identities:

We can use formulas from this section to prove identities

$$\sin(8x) = 2\sin(4x)\cos(4x)$$

$$\sin(8x) = \sin(2\underbrace{(4x)}_{\text{"x" in formula}})$$

$$= 2\sin(4x)\cos(4x)$$

reason

$\sin(8x) = \sin(2\hat{x})$

for $\hat{x} = 4x$

$\sin(2\hat{x}) = 2\sin\hat{x}\cos\hat{x}$

plug in $\hat{x}$

$2\sin(4x)\cos(4x)$

$$\frac{1+\sin(2x)}{\sin(2x)} = 1 + \frac{1}{2}\sec(x)\csc(x)$$

Work

$$\frac{1+\sin(2x)}{\sin(2x)} = \frac{1 + 2\sin x\cos x}{2\sin x\cos x}$$

$$= \frac{1}{2\sin x\cos x} + \frac{2\sin x\cos x}{2\sin x\cos x}$$

$$= \frac{1}{2}\cdot\frac{1}{\sin x}\cdot\frac{1}{\cos x} + 1$$

$$= \frac{1}{2}\csc x\sec x + 1$$

reason

$\sin(2x) = 2\sin\cos x$

fraction property

$\frac{1}{\sin x} = \csc x$

$\frac{1}{\cos x} = \sec x$

Quiz 7.1 Monday

7.4 and 7.5- Trigonometric Equations

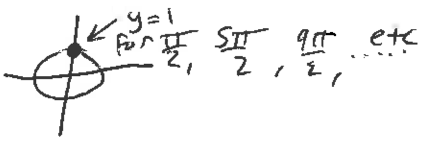
7.4 and 7.5 can difficult, yet fun sections. They can be difficult because they combine many different trig ideas that we have studied this semester. They are fun because this is your opportunity to use your brain in creative ways!

$(\theta = \sin^{-1} 1)$

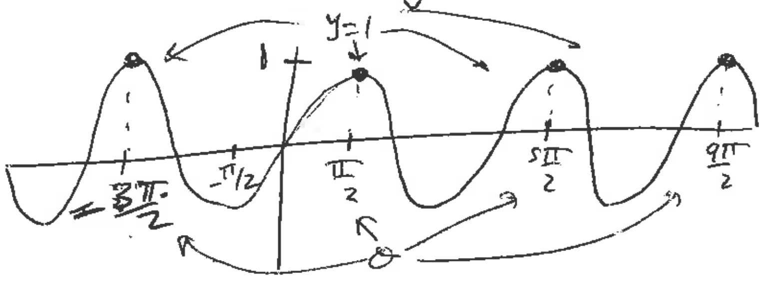
(really similar to ch 5.5 trig inveses)

Think: How could we solve the equation $\sin\theta = 1$?

Since $\sin\theta = y$, what θ value gives $y=1$?



Graph of sine:



Note In order for $\sin\theta = x$ and $\cos\theta = x$ to have solutions, we need $-1 \leq x \leq 1$. Otherwise, no solution exists

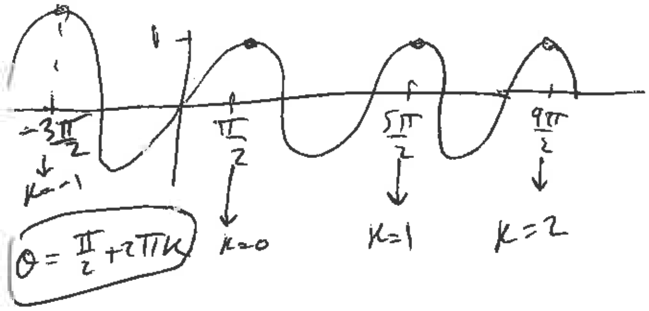
Infinitely many solutions

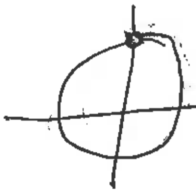
It is important to realize that if a trig equation has a single solution, then it has infinitely many solutions. Think about the unit circle and/or the graph of the trig functions.

So if we want $\sin\theta = 1$ we need $\theta = \frac{\pi}{2}$ or $\frac{\pi}{2} + 2\pi k$

where $k = 0, \pm 1, \pm 2, \ldots\ldots$

ie, $k \in \mathbb{Z}$ (k is an integer)



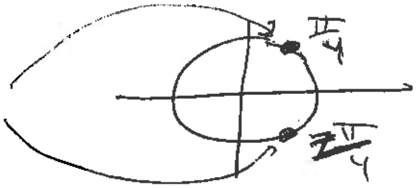


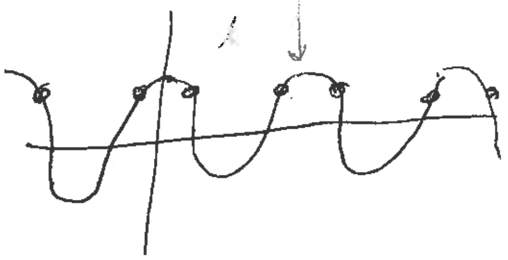
$\frac{\pi}{2} \pm 2\pi k$ puts us back at $y=1$ for all $k = 0, \pm 1, \ldots\ldots$

In webassign/tests you will be asked for multiple values that are solutions. Keep in mind the unit circle and where appropriate values occur. For example...

$\cos\theta = \frac{\sqrt{2}}{2}$.

$\cos\theta = x = \frac{\sqrt{2}}{2}$





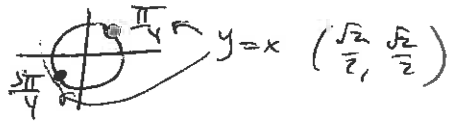
$\Rightarrow \theta = \frac{\pi}{4}$ or $\frac{7\pi}{4}$.

$\Rightarrow \boxed{\theta = \frac{\pi}{4} + 2\pi k \text{ or } \frac{7\pi}{4} + 2\pi k \text{ where } k \text{ is any integer}}$

Note: $\tan$, $\csc^2$, $\sec^2$, $\cot$ all will have $+\pi k$ instead of $\pm 2\pi k$ or $(\text{trig})^2$

Examples

$\tan\theta = 1$ => need $\frac{y}{x} = 1$ => $y = x$



$\theta = \frac{\pi}{4} + \pi k$ for $k = 0, \pm1, \pm2 \ldots$

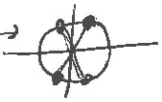
only need $+\pi k$ b/c distance btween the points are π, not 2π

Sometimes a trig function might be squared.

$4\sin^2\theta - 3 = 0$ → isolate $\sin\theta$

$4\sin^2\theta = 3 \rightarrow \sin^2\theta = \frac{3}{4} \rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2}$

where does $\sin\theta = y = \pm\frac{\sqrt{3}}{2}$ →



So we need

$\bar{\theta} = \frac{\pi}{3}$ => $\theta = \frac{\pi}{3} + \pi k$ or $\frac{2\pi}{3} + \pi k$

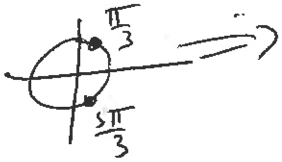
Sometimes you need to factor the equation

$4\cos^2\theta - 4\cos\theta + 1 = 0$

"$4x^2 - 4x + 1$" where $x = \cos\theta$

=> $(2\cos\theta - 1)(2\cos\theta - 1) = 0$

=> $\cos\theta = \frac{1}{2}$



$\theta = \frac{\pi}{3} + 2\pi k$

or

$\frac{5\pi}{3} + 2\pi k$

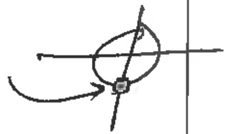
Sometimes you need to factor the equation

$\sin^2\theta - \sin\theta - 2 = 0$

"$x^2 - x - 2$"

$(\sin\theta - 2)(\sin\theta + 1) = 0$

$\sin\theta = 2$ or $\sin\theta = -1$



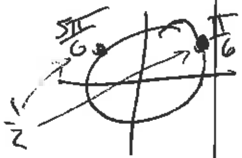
not possible since sin has maximis of ± 1

$\theta = \frac{3\pi}{2} + 2\pi k$ for $k \in \mathbb{Z}$

Sometimes you need to solve for the θ inside the function

$\sin 3\theta = \frac{1}{2}$

where does $\sin\theta = y = \frac{1}{2}$



=> we need $3\theta = \frac{5\pi}{6} + 2\pi k$

or

$3\theta = \frac{\pi}{6} + 2\pi k$

Solve for θ

$\theta = \frac{\pi}{18} + \frac{2\pi k}{3}$

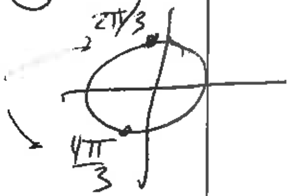
or

$\theta = \frac{\pi}{18} + \frac{2\pi k}{3}$

Sometimes you need to solve for the θ inside the function

$2\cos(2\theta) + 1 = 0$

$\cos(2\theta) = -\frac{1}{2}$



need $2\theta = \frac{2\pi}{3} + 2\pi k$

or $2\theta = \frac{4\pi}{3} + 2\pi k$

solve for θ

$\theta = \frac{2\pi}{6} + \frac{2\pi k}{2}$ or $\frac{4\pi}{6} + \frac{2\pi k}{2}$

$= \frac{\pi}{3} + \pi k$ or $= \frac{2\pi}{3} + \pi k$

Sometimes you have to use one of the following double angle formulas or identities

DOUBLE-ANGLE FORMULAS

Formula for sine: ① $\sin 2x = 2 \sin x \cos x$

Formulas for cosine: ② $\cos 2x = \cos^2 x - \sin^2 x$

③ $= 1 - 2 \sin^2 x$

④ $= 2 \cos^2 x - 1$

Formula for tangent: ⑤ $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

$$\boxed{\sin^2\theta + \cos^2\theta = 1} \Leftrightarrow \cos\theta = \sqrt{1-\sin^2\theta} \Leftrightarrow \sin\theta = \sqrt{1-\cos^2\theta}$$

$\cos 2\theta + \cos\theta = 2$

Use (4) => $2\cos^2 x - 1 + \cos\theta = 2$

=> $2\cos^2\theta + \cos\theta - 3 = 0$

=> $(2\cos\theta + 3)(\cos\theta - 1) = 0$

$2\cos\theta + 3 = 0$ or $\cos\theta - 1 = 0$

$\cos\theta = -\frac{3}{2}$ $\cos\theta = 1$

Not Possible

$\boxed{\theta = 0\pi + 2\pi k}$

$\sin^2\theta = 4 - 2\cos^2\theta$ → Sub in $\sin^2 = 1 - \cos^2$

$1 - \cos^2\theta = 4 - 2\cos^2\theta$

$\cos^2\theta - 3 = 0$

$\cos\theta = \sqrt{3} \approx 1.\ldots$

↳ Not possible since

$\cos \in [-1, 1]$

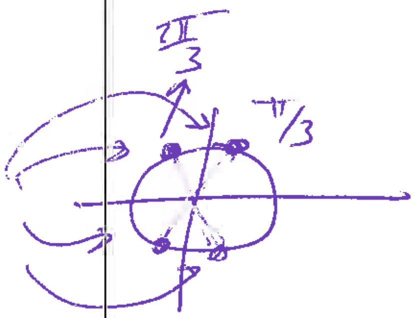
$2\sin^2\theta = 2 + \cos 2\theta$ → Sub in w/ ③

$2\sin^2\theta = 2 + (1 - 2\sin^2\theta)$

$4\sin^2\theta - 3 = 0$

$\sin\theta = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$

=> $\theta = \boxed{\begin{matrix}\frac{\pi}{3} + \pi k \\ \frac{2\pi}{3} + \pi k\end{matrix}}$



The following material found on this page is assumed knowledge that can be used as a reference and reminder. For more help reviewing fractions or any of the following properties see the information on pages 3-9 of the textbook and/or come see me.

PROPERTIES OF REAL NUMBERS

Property	Example	Description
Commutative Properties		
$a + b = b + a$	$7 + 3 = 3 + 7$	When we add two numbers, order doesn't matter.
$ab = ba$	$3 \cdot 5 = 5 \cdot 3$	When we multiply two numbers, order doesn't matter.
Associative Properties		
$(a + b) + c = a + (b + c)$	$(2 + 4) + 7 = 2 + (4 + 7)$	When we add three numbers, it doesn't matter which two we add first.
$(ab)c = a(bc)$	$(3 \cdot 7) \cdot 5 = 3 \cdot (7 \cdot 5)$	When we multiply three numbers, it doesn't matter which two we multiply first.
Distributive Property		
$a(b + c) = ab + ac$ $(b + c)a = ab + ac$	$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5$ $(3 + 5) \cdot 2 = 2 \cdot 3 + 2 \cdot 5$	When we multiply a number by a sum of two numbers, we get the same result as we would get if we multiply the number by each of the terms and then add the results.

PROPERTIES OF FRACTIONS

Property	Example	Description
1. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$	When **multiplying fractions**, multiply numerators and denominators.
2. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$	When **dividing fractions**, invert the divisor and multiply.
3. $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$	$\frac{2}{5} + \frac{7}{5} = \frac{2 + 7}{5} = \frac{9}{5}$	When **adding fractions** with the **same denominator**, add the numerators.
4. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$	$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 3 \cdot 5}{35} = \frac{29}{35}$	When **adding fractions** with **different denominators**, find a common denominator. Then add the numerators.
5. $\frac{ac}{bc} = \frac{a}{b}$	$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$	**Cancel** numbers that are **common factors** in numerator and denominator.
6. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$	$\frac{2}{3} = \frac{6}{9}$, so $2 \cdot 9 = 3 \cdot 6$	**Cross-multiply.**

Fractions are important
See black board document and/or me for help w/ them

PROPERTIES OF NEGATIVES

Property	Example
1. $(-1)a = -a$	$(-1)5 = -5$
2. $-(-a) = a$	$-(-5) = 5$
3. $(-a)b = a(-b) = -(ab)$	$(-5)7 = 5(-7) = -(5 \cdot 7)$
4. $(-a)(-b) = ab$	$(-4)(-3) = 4 \cdot 3$
5. $-(a + b) = -a - b$	$-(3 + 5) = -3 - 5$
6. $-(a - b) = b - a$	$-(5 - 8) = 8 - 5$

DEFINITION OF ABSOLUTE VALUE

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

PROPERTIES OF ABSOLUTE VALUE

Property	Example	Description
1. $\lvert a \rvert \geq 0$	$\lvert -3 \rvert = 3 \geq 0$	The absolute value of a number is always positive or zero.
2. $\lvert a \rvert = \lvert -a \rvert$	$\lvert 5 \rvert = \lvert -5 \rvert$	A number and its negative have the same absolute value.
3. $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	$\lvert -2 \cdot 5 \rvert = \lvert -2 \rvert \lvert 5 \rvert$	The absolute value of a product is the product of the absolute values.
4. $\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$	$\left\lvert \frac{12}{-3} \right\rvert = \frac{\lvert 12 \rvert}{\lvert -3 \rvert}$	The absolute value of a quotient is the quotient of the absolute values.

Types of Numbers

- **Natural Numbers** – $\{1, 2, 3, 4, \ldots\}$ – whole positive numbers – $\mathbb{N}$
- **Integers** – $\{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$ – whole positive & negative numbers – $\mathbb{Z}$
- **Rational Numbers** – any number that can be written as $\frac{a}{b}$ where a, b integers $\mathbb{Q}$
- **Irrational Numbers** – Not rational. have nonterminating and nonrepeating decimals $\boxed{ex}$ π, $\sqrt{2}$, e
- **Real Numbers** – Union of all the above – $\mathbb{R}$

Sets and Intervals

Set- a collection of elements

Two ways to represent a set:

1) List the elements between braces

$\boxed{ex}$ $A = \{1, 2, 3, 4\}$

2) Set-builder Notation

$\boxed{ex}$ $A = \{x \mid x \text{ is an integer and } 0<x<5\}$

"All x "such that" x is an integer between 0 and 5"

Notation:

"$\cup$" means union

$\boxed{ex}$ $A = \{1, 2, 3, 4\}$ $B = \{2, 4, 6, 8\}$

"$\cap$" means intersection

$A \cup B = \{1, 2, 3, 4, 6, 8\}$

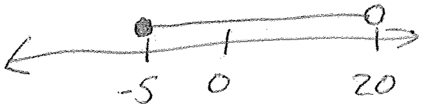
$A \cap B = \{2, 4\}$

Intervals – If a<b, the interval from a to b consists of all numbers between a and b

Notation: ( ) – open interval – do not include end points – ○

[ ] – Closed interval – do include end points – ●

$\boxed{ex}$ $[-5, 20)$



1.2- Exponents & Radicals

$\boxed{ex}$ $2^4 = 2\cdot2\cdot2\cdot2$

$(x+y)^2 = (x+y)(x+$

$\neq (x^2+y^2)$

EXPONENTIAL NOTATION

If a is any real number and n is a positive integer, then the **nth power** of a is

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

The number a is called the **base**, and n is called the **exponent**.

Properties of Exponents

ZERO AND NEGATIVE EXPONENTS

If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$

EXs:

$4^0 = 1$ $\quad 98246^0 = 1$ $\quad 4^{-1} = \frac{1}{4}$ $\quad x^{-3} = \frac{1}{x^3}$

$(x^2 + e^x + \ln y + 1000)^0 = 1$ $\quad \frac{1}{y^{-3}} = y^3$

LAWS OF EXPONENTS

Law	Example	Description
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^5 = 3^{2+5} = 3^7$	To multiply two powers of the same number, add the exponents.
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^5}{3^2} = 3^{5-2} = 3^3$	To divide two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	$(3^2)^5 = 3^{2\cdot5} = 3^{10}$	To raise a power to a new power, multiply the exponents.
4. $(ab)^n = a^n b^n$	$(3 \cdot 4)^2 = 3^2 \cdot 4^2$	To raise a product to a power, raise each factor to the power.
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$	To raise a quotient to a power, raise both numerator and denominator to the power.

need Multiplicati

$$(8a^2z)\left(\tfrac{1}{2}a^3z^4\right) = 8\cdot\tfrac{1}{2}\cdot a^2\cdot a^3\cdot z^1\cdot z^4$$

$$= 4\,a^{2+3}\,z^{1+4} = 4a^5z^5$$

$$\frac{(u^{-1}v^2)^2}{(u^3v^{-2})^3} = \frac{(u^{-1}v^2)(u^{-1}v^2)}{(u^3v^{-3})(u^3v^{-2})(u^3v^{-2})}$$

$$= \frac{u^{-1-1}\,v^{2+2}}{u^{3+3+3}\,v^{-2-2-2}} = \frac{u^{-2}v^4}{u^9v^{-6}} = u^{-2-9}\,v^{4-(-}$$

$$= u^{-11}v^{10} = \frac{v^{1}}{u^{1}}$$

$$\left(\frac{xy^{-2}z^{-3}}{x^2y^3z^{-4}}\right)^{-3} = \frac{(x)^{-3}(y^{-2})^{-3}(z^{-3})^{-3}}{(x^2)^{-3}(y^3)^{-3}(z^{-4})^{-3}}$$

$$= \frac{x^{-3}\,y^6\,z^9}{x^{-6}\,y^{-9}\,z^{12}} = x^{-3-(-6)}\,y^{6-(-9)}\,z^{9-12} = x^3y^{15}z^{-3} = \frac{x^3y^{15}}{z^3}$$

$$(2u^2v^3)^3(3u^{-3}v)^2$$

$$2^3(u^2)^3(v^3)^3\,3^2(u^{-3})^2\,v^2$$

$$8\,u^6\,v^9\,9\,u^{-6}\,v^2$$

$$8\cdot9\,u^{6+(-6)}\,v^{9+2} = 72\,\underbrace{u^0}_{=1}\,v^{11}$$

$$= 72\,v^{11}$$

Scientific Notation

Some numbers that humans encounter are too large or too small to write out in decimal form, which is why scientific notation was developed.

SCIENTIFIC NOTATION

A positive number x is said to be written in **scientific notation** if it is expressed as follows:

$$\underline{x = a \times 10^n} \qquad \text{where } 1 \leq a < 10 \text{ and } n \text{ is an integer}$$

Examples:

$$2.1 \underline{\times 10^6} = 2\underbrace{100000}. = 2,100,000$$

" Move the decimal 6 places to the right "

$$3 \times 10^{-4} = \underbrace{.0003} = .0003$$

To do arithmetic group like terms and use exponent properties

ex $$\frac{(3.9 \times 10^{-2})(2 \times 10^{3})}{(4.8 \times 10^{12})}$$

$$= \frac{(3.9)(2)}{(4.8)} \times 10^{-2+3-12} = \frac{7.8}{4.8} \times 10^{-11}$$

Radicals

Definition of square root:

$$\sqrt{a} = b \qquad \text{means} \qquad b^2 = a \qquad \text{and} \qquad b \geq 0$$

ex

$$\sqrt[3]{4} = 2 \quad \text{means} \quad 4 = 2^2$$

DEFINITION OF nth ROOT

If n is any positive integer, then the **principal nth root** of a is defined as follows:

$$\sqrt[n]{a} = b \qquad \text{means} \qquad b^n = a$$

If n is even, we must have $a \geq 0$ and $b \geq 0$.

$$3^3 = 27 \quad \text{means} \quad 3 = \sqrt[3]{27}$$

$$\sqrt[3]{8} = 2 \quad \text{means} \quad 8 = 2^3$$

PROPERTIES OF nth ROOTS

Property

1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$
4. $\sqrt[n]{a^n} = a$ if n is odd
5. $\sqrt[n]{a^n} = |a|$ if n is even

Examples

$$\sqrt[3]{8 \cdot 27} = \sqrt[3]{8} \cdot \sqrt[3]{27} = 2 \cdot 3 = 6$$

$$\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \text{[illegible]} \; \frac{4}{5}$$

$$\sqrt[4]{\sqrt[5]{2}} = \sqrt[4 \cdot 5]{2} = \sqrt[20]{2}$$

$$\sqrt[3]{-2^3} = -2$$

$$\sqrt[2]{-2^2} = |-2| = 2$$

Simplifying Radical Expressions- Method #1

One way to simplify a radical expression is to factor out the largest nth root and combine with like radicals

Note: need to keep in mind properties from previous page

Exs:

① $3\sqrt{7} - 4\sqrt{7} = (3-4)\sqrt{7} = -\sqrt{7}$

② $\sqrt{162} + \sqrt{50} = \sqrt{81 \cdot 2} + \sqrt{25 \cdot 2} = \sqrt{81}\cdot\sqrt{2} + \sqrt{25}\cdot\sqrt{2}$

$= 9\sqrt{2} + 5\sqrt{2} = 14\sqrt{2}$

③ $\sqrt[3]{8x^5} - \sqrt[3]{x^2} = \sqrt[3]{8}\,\sqrt[3]{x^3}\,\sqrt[3]{x^2} - \sqrt[3]{x^2} = 2x\sqrt[3]{x^2} - \sqrt[3]{x}$

$= (2x-1)\sqrt[3]{x^2}$

DEFINITION OF RATIONAL EXPONENTS

For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = (\sqrt[n]{a})^m \quad \xrightarrow{\text{Same}} \quad \text{or equivalently} \quad a^{m/n} = \sqrt[n]{a^m}$$

If n is even, then we require that $a \geq 0$.

$$\boxed{a^{1/n} = \sqrt[n]{a}}$$

Showing how to tra
between fraction
exponents and radicals

EXs:

① $a^{1/2} = \sqrt[2]{a}$, $\sqrt[3]{x} = x^{1/3}$, $\sqrt[10]{9} = 9^{1/10}$

$\sqrt{4} = 4^{1/2} = 2$

② $\sqrt[3]{8^2} = 8^{2/3} \longrightarrow$ Notice: $\sqrt[3]{8^2} = \sqrt[3]{64} = 4$

$(\sqrt[3]{8})^2 = (2)^2 = 4$

same

- $16^{-1/4} = \dfrac{1}{16^{1/4}} = \dfrac{1}{\sqrt[4]{16}} = \dfrac{1}{2}$

- $\dfrac{1}{\sqrt[2]{y^5}} = \dfrac{1}{y^{5/2}} = y^{-5/2}$

all this used on next page in Method #2

Simplifying Radical Expressions- Method #2

A second method to simplify a radical expression is to change the radical to an exponent and use the properties from page 1.

Exs: $2\cdot\sqrt[2]{y}\cdot 4\cdot\sqrt[4]{y} = 2\cdot 4\cdot y^{1/2}\cdot y^{1/4} = 8y^{1/2+1/4} = 8y^{\frac{2}{4}+\frac{1}{4}} = 8y^{3/4}$

$= 8\sqrt[4]{y^3}$

- $\sqrt[6]{a}\cdot\sqrt[5]{a} = a^{1/6}\cdot a^{1/5} = a^{1/5+1/6} = a^{6/30+5/30} = a^{11/30} = \sqrt[30]{a^{11}}$

- $\dfrac{x^{1/3}\cdot x^{2/3}}{x^{1/3}} = \dfrac{x^{1/3+2/3}}{x^{1/3}} = \dfrac{x^{3/3}}{x^{1/3}} \quad x^{3/3-1/3} = x^{2/3} = \sqrt[3]{x^2}$

- $\dfrac{(x^5z^{10})^{1/5}}{(x^3z^6)^{-1/3}} = \dfrac{(x^5)^{1/5}(z^{10})^{1/5}}{(x^3)^{-1/3}(z^6)^{-1/3}} = \dfrac{x^1z^2}{x^{-1}z^{-2}} = (x^1z^2)(x^1z^2)$

$6\cdot\frac{-1}{3} = \frac{-6}{3} = -2$

$= x^2z^4$

Rationalizing the Denominator

– ie, getting rid of a root in denominator

If a fraction has a radical in the denominator of the form $\sqrt[n]{a^m}$ we can rationalize it by multiplying both the numerator and denominator by $\sqrt[n]{a^{n-m}}$

Note: webassign requires denominator to be rationalized

Exs:

here, $n=5$, $m=2$

- $\dfrac{4}{\sqrt[5]{x^2}}$ – to get rid of root in denominator multiply by $\dfrac{\sqrt[5]{x^{5-2}}}{\sqrt[5]{x^{5-2}}} = \dfrac{\sqrt[5]{x^3}}{\sqrt[5]{x^3}}$

$\dfrac{4}{\sqrt[5]{x^2}}\cdot\dfrac{\sqrt[5]{x^3}}{\sqrt[5]{x^3}} = \dfrac{4}{x^{2/5}}\cdot\dfrac{x^{3/5}}{x^{3/5}} = \dfrac{4x^{3/5}}{x^{2/5+3/5}} = \dfrac{4x^{3/5}}{x^{5/5}} = \dfrac{4\sqrt[5]{x^3}}{x}$

- $\dfrac{2}{\sqrt[3]{y}}$ $n=3$, $m=1$

$\dfrac{2}{y^{1/3}}\cdot\dfrac{\sqrt[3]{y^{3-1}}}{\sqrt[3]{y^{3-1}}} = \dfrac{2y^{2/3}}{y^{1/3}\cdot y^{2/3}} = \dfrac{2y^{2/3}}{y} = \dfrac{2\sqrt[3]{}}{y}$

1.3- Algebraic Expressions

POLYNOMIALS

A **polynomial** in the variable x is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x^1 + a_0X^0$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a nonnegative integer. If $a_n \neq 0$, then the polynomial has **degree n**. The monomials a_kx^k that make up the polynomial are called the **terms** of the polynomial.

Ex: A degree 4 polynomial- $x^4 + 3x + 1$, $\pi x^4 - e$

NonEx: $\sqrt{x} + 4 = x^{1/2} + 4$, $\frac{y-2}{y+2}$, $e^x - \pi\sqrt{x} + \frac{1}{x}$

Adding and Subtracting Polynomials

Like terms: Terms with the same variable raised to the same power Ex $x^2 + 2x^2 = 3x^2$

When adding/subtracting two polynomials, you can only combine like terms

Ex: $(x^3 + 2x - 5) - (4x^3 - x^2 - 4x + 10)$

$-3x^3 + 6x + x^2 - 15$

Multiplying Algebraic Expressions

To find the product of two algebraic expressions you must use the **distributive property** to multiply each term of the first expression by each term of the second expression.

Ex: $(4y - 10)(6y^2 + 8y + 2)$ $= -60y^2 - 80y - 20 + 24y^3 + 32y^2 + 8y$

$= -28y^2 - 72y + 24y^3 - 20$

Ex: $(4x - 5y)(3x - y)$

$= -15xy + 5y^2 + 12x^2 - 4xy$

$= 5y^2 + 12x^2 - 19xy$

$3\sqrt[5]{2} - 2\sqrt[5]{2}$

"$3x - 2x$"

$1\sqrt[5]{2}$

$(A+B)(A-B) = A^2-B^2+AB-AB = A^2-B^2$

Special Product Formulas

SPECIAL PRODUCT FORMULAS

If A and B are any real numbers or algebraic expressions, then

1. $(A+B)(A-B) = \underline{A^2-B^2}$ Sum and product of same terms
2. $(A+B)^2 = A^2+2AB+B^2$ Square of a sum
3. $(A-B)^2 = A^2-2AB+B^2$ Square of a difference
4. $(A+B)^3 = A^3+3A^2B+3AB^2+B^3$ Cube of a sum
5. $(A-B)^3 = A^3-3A^2B+3AB^2-B^3$ Cube of a difference

$(A-B)^2$

Ex: $(4-5x)^2$ → use #3 $A=4$, $B=5x$

$= 4^2-2\cdot4\cdot5x+(5x)^2$

$= 16-40x+5^2x^2$

$= 16-40x+25x^2$

Ex: $(x^2+y)^3$ – Use #4 $A=x^2$, $B=y$

$= (x^2)^3+3(x^2)^2y+3x^2(y)^2+(y)^3$

$= x^6+3x^4y+3x^2y^2+y^3$

Ex: $(y^3+1)(y^3-1)$ – Use #1 $A=y^3$, $B=1$

$(y^3)^2-(1)^2 = y^6-1$

Ex: $(16+\sqrt{z})^2$ – Use #2 $A=16$, $B=\sqrt{z}$

$= (16)^2+2\cdot16\cdot\sqrt{z}+(\sqrt{z})^2$

$= 256+32\sqrt{z}+z$

$\sqrt{z} = z^{1/2}$

$(\sqrt{z})^2 = (z^{1/2})^2 =$

Factoring

The rest of the section is devoted to factoring polynomials. Factoring is the opposite of multiplying algebraic expressions. That is, **factoring** an expression is re-writing the expression as a product of simpler ones.

Approaches to Factoring:

1) Look for a common factor
2) "Trial and Error" (with 3 terms)
3) Use a Special Factoring Formula
4) Grouping Terms (with 4 or more terms)

→ each discussed in pages to follow

1) Looking for a common factor

If each term of a polynomial share a common factor you can divide each term by the common factor and put the common factor out in front of the resulting expression

Ex: $4x^2 + 2x - 8$

Common factor: 2

$2 \cdot (2x^2 + x - 4)$

Check: $2 \cdot 2x^2 + 2 \cdot x - 2 \cdot 4$

$4x^2 + 2x - 8$

Ex: $8x^4y^2 + 6x^3y^3 - 2xy^4$

Common factor: $2xy^2$

$2xy^2 \cdot (4x^3 + 3x^2y - y^2)$

2) Trial and Error — need 3 terms

Trinomial- a polynomial of the form $ax^2 + bx + c$ where *a, b and c* are any real number

Trial and error is a method used to factor a trinomial

To factor such a polynomial you have to think of two numbers (call them *r and s)* such that *r+s=b and rs=c.* Then, factor the expression using the form $(x + r)(x + s)$

Ex: $x^2 + x - 6$ ← $\boxed{a=1}$ →

$r = 3$ Notice $r \cdot s = -6$

$s = -2$ $r+s = 1$

$(x+r)(x+s) = (x+3)(x-2)$

Ex: $x^2 - 2x - 35$ $r = -7$, $s = 5$

$= (x-7)(x+5) = x^2 - 2x - 35$

If $a \neq 1$ then we need to find numbers *p, q, r and s* such that *pq=a, rs=c* and *ps + qr = b*. Then, factor the expression using the form $(px + r)(qx + s)$

Ex: $6x^2 + 7x - 5$

~~want ps+qr=7~~

~~try p=6, q=1~~

$\boxed{ps + qr \neq 7}$

Factors of -5: 5, -1

$s = -1$, $r = 5$

want $ps + qr = 7$

try $p = 6$, $q = 1$

this didn't work

try $r = 5$, $s = -1$

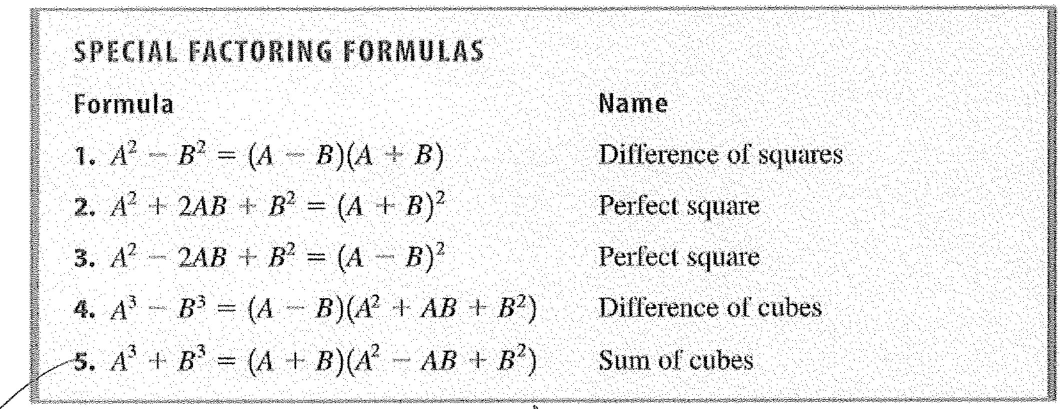
$p = 3$, $q = 2$

↳ these work

Since $pq = a$, $rs = c$ and $ps+qr=$

$(2x-1)(3x+5)$

3) Special Factoring Formulas



SPECIAL FACTORING FORMULAS

Formula	**Name**
1. $A^2 - B^2 = (A - B)(A + B)$	Difference of squares
2. $A^2 + 2AB + B^2 = (A + B)^2$	Perfect square
3. $A^2 - 2AB + B^2 = (A - B)^2$	Perfect square
4. $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$	Difference of cubes
5. $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$	Sum of cubes

these are helpful short cuts but Not always necessary

Ex: $(4+y)^3+8 = \underbrace{(4+y)^3+2^3}_{A^3+B^3}$

$A = 4+y$, $B=2$, use #5

$= ((4+y)+2)((4+y)^2-(4+y)2+2^2)$

$= (6+y)((4+y)^2-(8+2y)+4)$

Ex: $27 - x^3 = 3^3 - x^3$ → use #4 $A=3$, $B=X$

$= (3-x)(3^2+3x+x^2)$

$= (3-x)(9+3x+x^2)$

Ex $x^2+4x+4 = x^2+2\cdot 2\cdot x+2^2$

Use #2 w/ $A=x$, $B=2$

$= (A+B)^2 = (x+2)^2$

4) Grouping Terms

If a polynomial has at least 4 terms, first grouping terms with ① a common factor can be an effective approach. Then factor ② each grouping and look for a final ③ common factor. Note you may need to re-arrange the expression 1st

Ex: $\underbrace{x^3+x^2}_{\text{group A}}+\underbrace{4x+4}_{\text{group B}}$

①

② $x^2(x+1)+4(x+1)$

③ $\boxed{(x^2+4)(x+1)}$

Ex: $\underbrace{x^3+2x^2}_{\text{group A}}-\underbrace{6x-12}_{\text{group B}}$

①

② $x^2(x+2)-6(x+2)$

③ $\boxed{(x^2-6)(x+2)}$

1.4- Rational Expressions

$2 = 2x^0$

Domain of an Algebraic Expression

Rational Expression- a fractional expression where both the numerator as well as the denominator are polynomials

all polynomials

Exs: $\frac{2}{3}$ $\frac{2x}{x-1}$ $\frac{x}{x^2+1}$ $\frac{x^3-x}{x^2-5x+6}$

$\boxed{\text{Nonex}}$ $\frac{\sqrt{x}}{e^x}$ → Not polynomials

Domain of an Algebraic Expression- All real numbers that can be plugged into the given variable that will result in a sound expression.

To find out what the domain of an expression is consider the real numbers that, when plugged into the expression, will result in an undefined value.

Factor wherever possible to help determine the domain

We write the domain in set-builder notation – $\{X \mid a < x \leq b\}$

Possible problems: **a)** Dividing by 0 **b)** Taking the root of a negative number **c)** Both **a** and **b**

Ex: $\frac{x}{(x^2-4)} = \frac{X}{(X+2)(X-2)}$

If $X = 2, -2$ the expression is undefined so we can't include $2, -2$ in domain

Domain: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

$\{X \mid X \neq 2 \text{ and } X \neq -2\}$

Ex: $\frac{\sqrt{r}}{(r+9)(r-12)}$

$r \neq -9$, $r \neq 12$ from denominator

$r \geq 0$ from numerator (so we don't take root of a negative)

Domain: $[0,12)\cup(12,\infty)$

$\{X \mid r \geq 0 \text{ and } r \neq 12\}$

Ex: $y^2 + 4y - 10$

No possible problems

Domain: $(-\infty, \infty)$

$\{y \mid y \text{ is a real number}\}$

Polynomials have the domain of all real Numbers

Simplifying Rational Expressions

In order to simplify rational expressions you can factor the numerator and denominator and cancel using the following property:

$$\boxed{\frac{A\cancel{C}}{B\cancel{C}} = \frac{A}{B}}$$

$\frac{A+C}{B+C} \neq \frac{A}{B}$

Ex: $\frac{x+2}{(x^2-6x-16)}$

$= \frac{\cancel{X+2}}{(\cancel{X+2})(X-8)}$ $\left(\frac{\cancel{A}}{\cancel{A}B}\right)$

$= \frac{1}{X-8}$

use original

Domain: $\{X \mid X \neq -2, X \neq 8\}$

Ex: $\frac{x^2-2x-3}{(x^2+6x+5)}$

$\frac{(X-3)(\cancel{X+1})}{(X+5)(\cancel{X+1})}$ $\left(\frac{A\cancel{B}}{C\cancel{B}}\right)$

$= \frac{(X-3)}{(X+5)}$

Domain: $(-\infty,-5)\cup(-5,-1)\cup(-1,\infty$

Multiplying Rational Expressions

When two rational expressions are being multiplied use the following property:

$$\frac{A}{B}\cdot\frac{C}{D}=\frac{AC}{BD}$$

ex $\frac{2}{3}\cdot\frac{3}{5}=\frac{2\cdot 3}{3\cdot 5}=\frac{6}{15}$

Ex: $\frac{P-6}{(P^2+2P-1)}\cdot\frac{4+P^2}{P}=\frac{(P-6)(4+P^2)}{(P^2+2P-1)\cdot P}=\frac{4P-24+P^3-6P^2}{P^3+2P^2-P}$

Dividing Rational Expressions

When dividing one rational expression by another use the following property combined with the above property of multiplication

$\frac{\frac{A}{B}}{\frac{C}{D}}=\frac{A}{B}\cdot\frac{D}{C}$

That is, use $\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}$ followed by $\frac{A}{B}\cdot\frac{C}{D}=\frac{AC}{BD}$

ex $\frac{2}{3}\div\frac{3}{5}=\frac{2}{3}\cdot\frac{5}{3}=\frac{10}{9}$

Ex: $\frac{x-2}{x^2+4}\div\frac{x^3+1}{x+6}$

$=\frac{(x-2)(x+6)}{(x^2+4)\cdot(x^3+1)}$

$=\frac{x^2+6x-2x-12}{x^5+4x^3+x^2+4}$

$=\frac{x^2+4x-12}{x^5+4x^3+x^2+4}$

Another?

Ex: $\frac{z+10}{1-z^2}\div\frac{10+z}{z}$

$\frac{z+10}{1-z^2}\cdot\frac{z}{10+z}=\frac{\cancel{(z+10)}\cdot z}{(1-z^2)\cdot\cancel{(10+z)}}$ $\left(\frac{\cancel{A}B}{C\cancel{A}}\right.$

$=\frac{z}{1-z^2}$

lesson: rational Expressions are just like fractions

Adding and Subtracting Rational Expression

- In order to add or subtract two rational expressions, like any fraction, they must have the same denominator.
- To write each term with the same denominator you must determine a common multiple of both denomiators.
- The easiest way to find a common multiple is to factor each denominator and take the product of the distinct factors. That is, multiply together each non-repeated factor of each denominator.
- Once you have determined the common multiple, multiply each rational expression appropriately by a form of 1 to reach a common denominator.
- Once both expressions have a common denominator, add/subtract the numerators while keeping the denominator the same

Ex: $\frac{4}{x-3} - \frac{x}{x+7}$ Common denominator: $(x-3)(x+7)$

$$\frac{4}{(x-3)}\cdot\frac{(x+7)}{(x+7)} - \frac{x}{(x+7)}\cdot\frac{(x-3)}{(x-3)} = \frac{4x+28}{(x-3)(x+7)} - \frac{x^2-3x}{(x-3)(x+7)}$$

Common denominator

$$= \frac{4x+28-x^2+3x}{(x-3)(x+7)} = \frac{-x^2+7x+28}{(x-3)(x+7)}$$

Ex: $\frac{1}{t^2-4} + \frac{t}{(t-2)^2} = \frac{1}{(t+2)(t-2)} + \frac{t}{(t-2)(t-2)}$

Common denominator:
$(t+2)(t-2)(t-2)$
$= (t+2)(t-2)^2$

$$\frac{1}{(t+2)(t-2)}\cdot\frac{(t-2)}{(t-2)} + \frac{t}{(t-2)(t-2)}\cdot\frac{(t+2)}{(t+2)}$$

$$= \frac{t-2}{(t+2)(t-2)^2} + \frac{t^2+2t}{(t+2)(t-2)^2} = \frac{t^2+3t-2}{(t+2)(t-2)^2}$$

Another? $\frac{2}{x} - \frac{3}{x-1} - \frac{4}{x^2-x} = \frac{2}{x} - \frac{3}{x-1} - \frac{4}{x(x-1)}$ Common Denom: $x(x-1)$

$$\frac{2}{x}\cdot\frac{(x-1)}{(x-1)\cancel{x}} - \frac{3}{x-1}\cdot\frac{x}{x} - \frac{4}{x(x-1)} = \frac{2x-2-3x-4}{x(x-1)} = \frac{-x-6}{x(x-1)}$$

Compound Fractions- A fraction where the numerator, the denominator or both are themselves fractional expressions

2 methods: **a)** Combine the numerator and/or denominator and then invert the fraction

easier, faster method → **b)** Multiply the numerator and ~~or~~ denominator by a common factor (common denominator)

$$\frac{x+\frac{1}{x+2}}{x-\frac{1}{x+2}}$$

Method a: Common denominator:

$$\frac{x\frac{(x+2)}{(x+2)}+\frac{1}{x+2}}{x\frac{(x+2)}{(x+2)}-\frac{1}{x+2}}=\frac{\frac{x^2+2x}{x+2}+\frac{1}{x+2}}{\frac{x^2+2x}{x+2}-\frac{1}{x+2}}=\frac{\frac{x^2+2x+1}{x+2}}{\frac{x^2+2x-1}{x+2}}$$

flip & multiply

$$=\frac{x^2+2x+1}{\cancel{x+2}}\cdot\frac{\cancel{x+2}}{x^2+2x-}$$

$$=\boxed{\frac{x^2+2x+1}{x^2+2x-1}}$$

Same

Method b: Common denominator: x+2

$$\frac{x+\frac{1}{x+2}}{x-\frac{1}{x+2}}\cdot\frac{x+2}{x+2}=\frac{x(x+2)+\frac{1}{\cancel{x+2}}\cdot\cancel{(x+2)}}{x(x+2)-\frac{1}{\cancel{x+2}}\cdot\cancel{(x+2)}}=\boxed{\frac{x^2+2x+1}{x^2+2x-1}}$$

Ex: $$\frac{\frac{t}{p}-\frac{p}{t}}{\frac{1}{t^2}+\frac{1}{p^2}}\cdot\frac{t^2p^2}{t^2p^2}=\frac{\frac{t}{\cancel{p}}\cdot\frac{t^2p^{\cancel{2}}}{\cancel{1}}-\frac{p}{\cancel{t}}\cdot\frac{t^{\cancel{2}}p^2}{1}}{\frac{1}{\cancel{t^2}}\cdot\frac{\cancel{t^2}p^2}{1}+\frac{1}{\cancel{p^2}}\cdot\frac{t^2\cancel{p^2}}{1}}=\boxed{\frac{t^3p-p^3t}{p^2+t^2}}$$

↓

Common denominator: t^2p^2

Rationalizing the Denominator or the Numerator

$\sqrt[3]{\ }$

If the numerator or denominator of a fraction has the form $\boxed{A + B\sqrt{C}}$ we can rationalize the numerator or denominator by multiplying the fraction by $\frac{A-B\sqrt{C}}{A-B\sqrt{C}}$ (called the conjugate radical)

If the numerator or denominator of a fraction has the form $\boxed{A - B\sqrt{C}}$ we can rationalize the numerator or denominator by multiplying the fraction by $\frac{A+B\sqrt{C}}{A+B\sqrt{C}}$

We are able to do this because of the special product formulas from section 1.3. Using these formulas we do not have to do any of the algebra

Rationalize the denominator

Ex: $\frac{5}{2+\sqrt{5}} \cdot \frac{2-\sqrt{5}}{2-\sqrt{5}}$

$$= \frac{10-5\sqrt{5}}{2^2-(\sqrt{5})^2}$$

$$= \frac{10-5\sqrt{5}}{4-5} = \frac{10-5\sqrt{5}}{-1}$$

$$= -10+5\sqrt{5}$$

Rationalize the numerator

Ex: $\frac{2\sqrt{3}-4}{x} = \frac{-4+2\sqrt{3}}{x} \cdot \frac{-4-2\sqrt{3}}{-4-2\sqrt{3}}$

$$= \frac{(-4)^2-(2\sqrt{3})^2}{-4x-2x\sqrt{3}}$$

$$= \frac{16-4\cdot 3}{-4x-2x\sqrt{3}} = \boxed{\frac{4}{-4x-2x\sqrt{3}}}$$

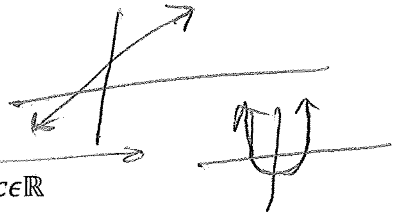
Don't make these common mistakes:

Correct multiplication property	Common error with addition
$(a \cdot b)^2 = a^2 \cdot b^2$	$(a + b)^2 \neq a^2 + b^2$
$\sqrt{a \cdot b} = \sqrt{a}\sqrt{b} \quad (a, b \geq 0)$	$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$
$\sqrt{a^2 \cdot b^2} = a \cdot b \quad (a, b \geq 0)$	$\sqrt{a^2 + b^2} \neq a + b$
$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$	$\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a + b}$
$\frac{\cancel{a}b}{1\cdot\cancel{a}} = \frac{b}{1} = b$	$\frac{a + b}{a} \neq b$
$a^{-1} \cdot b^{-1} = (a \cdot b)^{-1}$	$a^{-1} + b^{-1} \neq (a + b)^{-1}$

$\cancel{\frac{a+b}{a+b}}$

→ lesson: addition screws everything op

1.5- Equations



Linear Equation- an equation of the form $ax + b = 0$ where $a, b \epsilon \mathbb{R}$

Quadratic Equation- an equation of the form $ax^2 + bx + c = 0$ where $a, b, c \epsilon \mathbb{R}$

In order to solve an equation you must isolate the variable on one side of the equal sign. That is, you have to "get the variable by itself."

The general method for solving equations is to reverse the order of operations. When doing so, as long as you carry out the same operation on both sides of the equal sign you will preserve the equality and get the correct answer.

Key Facts: **1)** If $\sqrt{x} = y\ then\ x = y^2$ **2)** If $x = y^2 then\ y = \pm\sqrt{x}$

we get 2 answers when we take a square root

$\sqrt{x} = 4$ then $x = 4^2 = 16$

~~$\sqrt{-16} \neq 4$~~

$16 = y^2$ then $y = \pm\sqrt{16} = \pm 4$

$y = 4$ or $y = -4$

Solving Linear Equations

To solve a linear equation you must get all variables on one side of the equal sign and all numbers on the other side and then multiply/divide as needed to isolate the variable.

Try this one yourself: $2x + 8 = 6 - 4x$

$+4x \qquad +4x$

$6x + 8 = 6$

$-8 \qquad -8$

$\frac{\cancel{6}x}{\cancel{6}} = \frac{-2}{6} \longrightarrow \boxed{x = \frac{-2}{6} = -\frac{1}{3}}$

Solving Quadratic Equations (ie, equations w/ x^2 in them)

Methods: 1) Take the square root

2) Complete the square

3) Quadratic Formula

} See in pages to follow

1) If the quadratic equation is simple enough you may just need to take the $\boxed{\text{square root of both sides}}$

Ex: $\sqrt{y^2} = \sqrt{81}$

$y = \pm\sqrt{81} = \pm 9$

$y = 9, -9$

Ex:

Ex: $(y+4)^2 = 10$

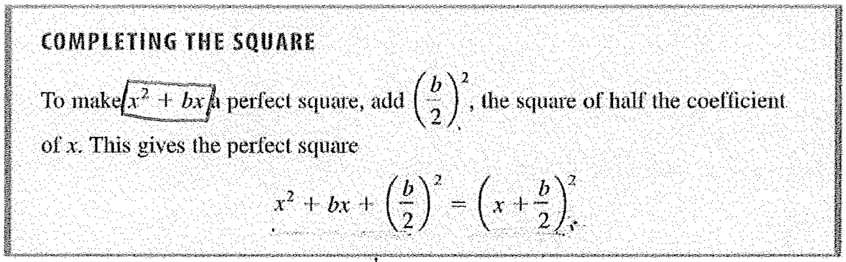
$\sqrt{(y+4)^2} = \pm\sqrt{10}$

$y+4 = \pm\sqrt{10}$

$-4 \qquad -4$

$\boxed{y = \underbrace{-4+\sqrt{10}}_{\text{larger}} \quad \text{or} \quad y = \underbrace{-4-\sqrt{10}}_{\text{smaller}}}$

2)



COMPLETING THE SQUARE

To make $\boxed{x^2 + bx}$ a perfect square, add $\left(\frac{b}{2}\right)^2$, the square of half the coefficient of x. This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Note: Need to have the equation in the form $\boxed{x^2+bx = \text{something}}$

Ex: $2v^2 - 6v + 4 = 0$

① we it in the form $v^2+bv = \#$

$2v^2 - 6v = -4$

$\frac{2(v^2-3v)}{2} = \frac{-4}{2}$

$v^2 - 3v = -2$ → correct form

$b = -3$

② add $\left(\frac{b}{2}\right)^2 = \left(\frac{-3}{2}\right)^2 = \frac{9}{4}$ to both sides

$v^2 - 3v + \frac{9}{4} = \underbrace{-2 + \frac{9}{4}}_{=1/4}$

~~One more?~~

$\underbrace{v^2 - 3v + \frac{9}{4}} = 1/4$

↓

$(v+(-3/2))^2 = 1/4$

⑤ take root

$v - 3/2 = \pm\sqrt{1/4}$

$v = \frac{3}{2} \pm 1/2 \Longrightarrow \boxed{v = 2, 1}$

Ex: $y^2 + 3y - 12 = 0$

① $y^2 + 3y = 12$ → correct form

② add $\left(\frac{b}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$ to both sides

$y^2 + 3y + \frac{9}{4} = 12 + \frac{9}{4}$

↓ by formula above

③ $\left(y + \frac{3}{2}\right)^2 = \frac{57}{4}$ → Solve like we did above

$y + \frac{3}{2} = \pm\sqrt{\frac{57}{4}}$

$-\frac{3}{2} \qquad -\frac{3}{2}$

$\boxed{y = -\frac{3}{2} \pm \frac{\sqrt{57}}{2}}$

3) Quadratic Formula- Can be used to solve any quadratic equation

THE QUADRATIC FORMULA

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ex: $\overset{a}{2}v^2 \overset{b}{-6}v + \overset{c}{4} = 0$

$X = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 2 \cdot 4}}{2(2)}$

$X = \frac{6 \pm \sqrt{36-32}}{4}$

Ex:

$X = \frac{6+\sqrt{4}}{4} = 2$ and $X = \frac{6-\sqrt{4}}{4} = 1$

Ex: $y^2 + 3y - 12 = 0$ → $a = 1$, $b = 3$, $C = -12$

$y = \frac{-3 \pm \sqrt{3^2 - 4(1)(-12)}}{2(1)}$

$y = \frac{-3 \pm \sqrt{57}}{2}$

$y = \frac{-3+\sqrt{57}}{2}$, $y = \frac{-3-\sqrt{57}}{2}$

$= -\frac{3}{2} + \frac{\sqrt{57}}{2}$ or $-\frac{3}{2} - \frac{\sqrt{57}}{2}$

Discriminant

↓

Note b^2-4ac is under the square root above in Quad eqn

THE DISCRIMINANT

The **discriminant** of the general quadratic $ax^2 + bx + c = 0$ $(a \neq 0)$ is $D = b^2 - 4ac$.

1. If $D > 0$, then the equation has two distinct real solutions.
2. If $D = 0$, then the equation has exactly one real solution.
3. If $D < 0$, then the equation has no real solution.

Ex: $2v^2 - 6v + 4 = 0$

$D = (-6)^2 - 4(2)4$

$D = 4 > 0$

$\Rightarrow 2v^2 - 6v + 4$

Ex:

has 2 distinct roots

Ex: $y^2 + 3y - 12 = 0$

$D = 3^2 - 4 \cdot 1(-12) = 57 > 0$

$\Rightarrow y^2 + 2y - 12 = 0$ has 2 distinct roots

$\boxed{ex}$ $2x^2 + 2x + 3 = 0$

$D = 2^2 - 4(2)(3) = -20 < 0$

$\Rightarrow 2x^2 + 2x + 3 = 0$ has no real Solution

Solving for a variable in terms of other variables

You can use the same steps as above to solve an equation for a given variable

Ex: solve the following equation for the variable m: $r^2 G\frac{mV}{\cancel{r^2}} = F\ r^2$

$$\Rightarrow \frac{\cancel{G}m\cancel{V}}{\cancel{G}\cancel{V}} = \frac{Fr^2}{GV} \Rightarrow m = \frac{Fr^2}{GV}$$

In webassign

$$\frac{1}{R} = \frac{1}{R_1} - \frac{1}{R_2}$$

Solve for R_1

① Isolate $\frac{1}{R_1}$ on one side

② add any fractions

③ Cross multiply to Solve for R_1

Solving an equation with a fractional expression

- ① First multiply both equations by the least common denominator. Doing so will eliminate all fractional expression(s).
- ② Once you have eliminated the fractional expression(s) you can solve the equation using one of the methods above

Ex: $\boxed{\frac{1}{x-1} - \frac{2}{x} = 1}$ → Common denominator: $(x-1)x$

$$\frac{1}{\cancel{x-1}}\cancel{(x-1)}x - \frac{2}{\cancel{x}}\cdot(x-1)\cdot\cancel{x} = 1\cdot(x-1)x$$

$$x - 2(x-1) = x^2 - x$$

$$x - 2x - 2 = x^2 - x$$

$\boxed{x^2 - [illegible]x + 2 = 0}$

↳ Solve using any Method

Solving an equation with a radical expression

① First isolate the radical.

② Second eliminate the radical by raising both sides of the equation to the same power.

③ Finally, solve the equation as you would any other.

Important: Not all answers you obtain from methods on this page are solutions. You must plug in your values to the original to check their validity (See 1.5 #18, 19) webassign

Ex: $2x + \sqrt{x+1} = 8$

$-2x \qquad -2x$

① $\sqrt{x+1} = 8 - 2x$

② $(\sqrt{x+1})^2 = \underbrace{(8-2x)^2}_{(8-2x)(8-2x)}$

$x + 1 = 4x^2 - 32x + 64$

③ $4x^2 - 33x + 63 = 0$

Quad formula: $\frac{33 \pm \sqrt{33^2 - 4(4)(63)}}{2(4)} \Rightarrow \boxed{x = 3 \text{ and } x = 21/4}$ → check then in original

$\underline{x=3}$

$2(3) + \sqrt{4} = 6 + 2 = 8$ ✓

it worked

$\underline{x = 21/4}$

$2(21/4) + \sqrt{21/4}$

$= \frac{42}{4} + \frac{\sqrt{21}}{4} \neq 8$

Since $\neq 8$ don't include $x = 21/4$

$\boxed{ex}$ $|x| = 4 \Rightarrow x = 4$, or $x = -4$

Solving an equation with an absolute value

Property: If $|x| = a$ then $x = a$ or $x = -a$

$|14 - x| = 28$

Use the property above to rewrite as

$14 - x = 28$ or $14 - x = -28$

$+x \quad +x \qquad +x \quad +x$

$-28 \quad -28 \qquad +28 \quad +28$

$\boxed{x = -14 \quad \text{or} \quad x = 42}$

check:

$|14 - (-14)| = |14 + 14| = |28| = 28$ ✓

$|14 - 42| = |-28| = 28$ ✓

$1 = |x - 6|$

$x - 6 = +1$ or $x - 6 = -1$

$+6 \quad +6 \qquad +6 \quad +6$

$\boxed{x = 7 \quad \text{or} \quad x = 5}$

~~$|8-6| = |-1|$~~ 1

$|7 - 6| = |1| = 1$

$|5 - 6| = |-1| = 1$

✓ They work

2013

1.7- Inequalities

RULES FOR INEQUALITIES

Rule	**Description**
1. $A \le B \iff A + C \le B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \le B \iff A - C \le B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \le B \iff CA \le CB$	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \le B \iff CA \ge CB$	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \le B \iff \frac{1}{A} \ge \frac{1}{B}$ $2 \le 3$ $\frac{1}{2} \ge \frac{1}{3}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \le B$ and $C \le D$, then $A + C \le B + D$	Inequalities can be added.

Make sure you know and understand #3, #4 and #5

Linear Inequalities

If an inequality is linear (the highest power of any variable is 1), you can use algebra and the above rules to isolate variable. Just make sure every operation you do you carry it out on all parts of the inequality.

Solve and graph: goal: get the variable by itself in the center of the inequality

$8 < -5x + 3 \le 18$

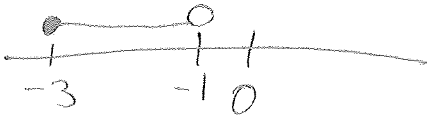
$-3 \quad -3 \quad -3$

$\frac{5}{-5} < \frac{-5x}{-5} \le \frac{15}{-5}$

switched by #4 above

$-1 > x \ge -3$

Interval: x in $[-3, -1)$



$2x \ge 8x - 6 > 2x + 12$

$-2x \quad -2x \quad -2x$

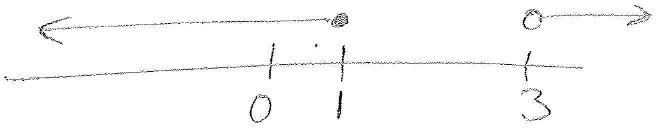
$0 \ge 6x - 6 > 12$

$+6 \quad +6 \quad +6$

$\frac{6}{6} \ge \frac{6x}{6} > \frac{18}{6}$

$1 \ge x > 3 \rightarrow x \le 1$ or $x > 3$

x in $(-\infty, 1] \cup (3, \infty)$



X^2 or greater

Nonlinear Inequalities

GUIDELINES FOR SOLVING NONLINEAR INEQUALITIES

1. **Move All Terms to One Side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the Intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals that are determined by these numbers.
4. **Make a Table or Diagram.** Use test values to make a table or diagram of the signs of each factor on each interval. In the last row of the table determine the sign of the product (or quotient) of these factors.
5. **Solve.** Determine the solution of the inequality from the last row of the sign table. Be sure to check whether the inequality is satisfied by some or all of the endpoints of the intervals. (This may happen if the inequality involves ≤ or ≥.)

→ Test values in each interval to see if they meet the criteria

$T^2 > 3(T+6)$

① $T^2 > 3T+18$

$T^2 - 3T - 18 > 0$

② $\underbrace{(T-6)}_{A}\underbrace{(T+3)}_{B} > 0$

③ $A=0$ if $\boxed{T=6}$ → These points are where the graph $=0$

$B=0$ if $\boxed{T=-3}$

$(-\infty,-3), (-3,6), (6,\infty)$

↳ on these intervals the graph is always positive or negative

↳ Test numbers in those intervals

Test: $-4, 0, 7$ → in the above intervals

$T=-4$: $(-4-6)(-4+3) = (-10)(-1) = 10 > 0$

$T=0$: $(-6)(3) = -18 < 0$

$T=7$: $(7-6)(7+3) = (1)(10) = 10 > 0$

based on

Another?

Result: $T^2 - 3T - 18 > 0$

on the interval

$(-\infty,-3) \cup (6,\infty)$

$(\overset{A}{y-3})(\overset{B}{y+6})^2 < 0$ → ≤ 0

①, ② already done

③ $A=0$ if $y=3$
$B=0$ if $y=-6$ → intervals are

$(-\infty,-6), (-6,3), (3,\infty$

Pick points in each interval: -7, 0, 10

④ $y=-7$: $(-7-3)(-7+6)^2 = (-10)(-1)^2 = -10 < 0$

$y=0$: $(-3)(6)^2 = -108 < 0$

$y=10$: $(10-3)(10+6)^2 = 7\cdot16^2 > 0$

⑤ $(y-3)(y+6)^2 < 0$

on the interval $(-\infty,-6) \cup (-6,3$

Set builder notation: $\{y \mid y<3 \text{ and } y \ne -6\}$

↳ don't include -6 since that makes original $=0$ want

Solving an Inequality Involving a Quotient

Follow the exact same steps as above

The main difference is that you need to factor both the numerator as well as the denominator and consider all factors of both in order to create your intervals.

* Note: Any interval end-point found in the denominator will **NOT** satisfy the inequality i.e. don't include then in the final answer

$$\frac{2x+1}{x-5} \le 3$$

① $\frac{2x+1}{x-5} - 3 \le 0 \rightarrow \frac{2x+1}{x-5} - \frac{3(x-5)}{x-5} \le 0$

② $\frac{-x+16}{x-5} \le 0$

③ Numerator = 0 when $\boxed{x=16}$ ← include

denominator = 0 when $\boxed{x=5}$

Intervals: $(-\infty,5), (5,16), (16,\infty)$

④ Test values: 0 10 20

$x=0$: $\frac{16}{-5} < 0$ ←

$x=10$: $\frac{-10+16}{10-5} = \frac{6}{5} > 0$

$x=20$:

$\frac{-20+16}{20-5} = \frac{-4}{15} < 0$ ←

⑤ $\frac{-x+16}{x-5} \le 0$ on the interval

$(-\infty,5) \cup [16,\infty)$

based on note above

≤ 0

$$\frac{s+1}{s-3} > -2$$

① $\frac{s+1}{s-3} + 2 > 0 \rightarrow \frac{s+1}{s-3} + \frac{2(s-3)}{s-3} > 0$

② $\boxed{\frac{3s-5}{s-3} > 0}$

③ Numerator = 0 when $\boxed{s=5/3}$

denominator = 0 when $\boxed{s=3}$

Intervals $(-\infty,5/3) \cup (5/3,3) \cup (3,$

④ Test values: 0 2 4

$s=0$: $\frac{-5}{-3} = \frac{5}{3} > 0$ ←

$s=2$: $\frac{6-5}{2-3} = -\frac{1}{1} = -1 < 0$

$s=4$: $\frac{12-5}{4-3} = \frac{7}{1} = 7 > 0$ ←

⑤ $\frac{3s-5}{s-3} > 0$

on interval $(-\infty,5/3) \cup (3,\infty)$

$|\cancel{x}| > 2$ $x > 2$ or $x < -2$

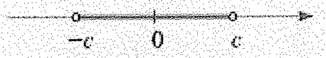
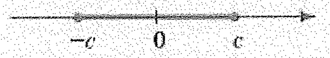
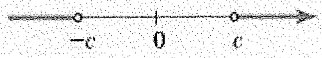
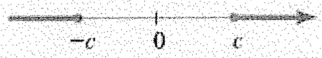
Absolute Value Inequalities

$\boxed{ex}$

$|\pm 6| < 7$

$-76 < 7$

PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

Inequality	Equivalent form	Graph
1. $\lvert x\rvert < c$	$-c < x < c$	
2. $\lvert x\rvert \le c$	$-c \le x \le c$	
3. $\lvert x\rvert > c$	$x < -c$ or $c < x$	
4. $\lvert x\rvert \ge c$	$x \le -c$ or $c \le x$	

$<$
$\le$

$>$
$\ge$

$|4x + 6| \le 18$

use #2

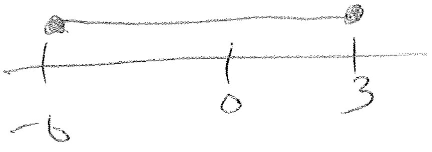
$-18 \le 4x+6 \le 18$
$-6 \quad -6 \quad -6$

$-24 \le 4x \le 12$
$\overline{4} \quad \overline{4} \quad \overline{4}$

$-6 \le x \le 3$

$x \in [-6, 3]$

past topics



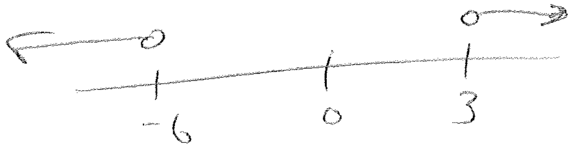
$18 < |4x + 6|$ $\xrightarrow{\text{rewrite}}$ $|4x+6| > 18$

use #3

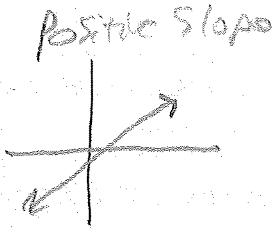
$4x+6 > 18$ or $4x+6 < -18$

$4x > 12$ or $4x < -24$

$\boxed{x > 3 \text{ or } x < -6}$



1.10- Lines

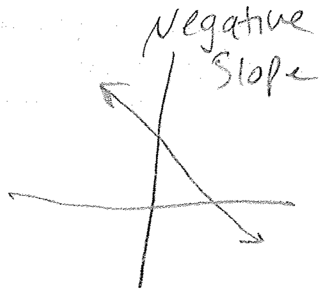


SLOPE OF A LINE

The **slope** m of a nonvertical line that passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \overset{\text{give graph}}{\boxed{\frac{\text{rise}}{\text{run}}}} = \boxed{\frac{y_2 - y_1}{x_2 - x_1}} \rightarrow \text{give 2 points}$$

The slope of a vertical line is not defined.



3 Types of Equations of a Line

1)

POINT-SLOPE FORM OF THE EQUATION OF A LINE

An equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y - \underset{\uparrow}{y_1} = \underset{\uparrow}{m}(x - \underset{\uparrow}{x_1})$$

Need: a point and a slope

2)

SLOPE-INTERCEPT FORM OF THE EQUATION OF A LINE

An equation of the line that has slope m and y-intercept b is

$$y = \underset{\uparrow}{m}x + \underset{\uparrow}{b}$$

Need: slope and y-intercept

Sometimes need to use slope and point to solve for y-intercep

3)

GENERAL EQUATION OF A LINE

The graph of every **linear equation**

$$\underset{\uparrow}{A}x + \underset{\uparrow}{B}y + \underset{\uparrow}{C} = 0 \qquad (A, B \text{ not both zero})$$

is a line. Conversely, every line is the graph of a linear equation.

→ rewrite slope intercept to get this form

Every non-vertical and non-horizontal line can be written in each of the previous forms

↓ ↓

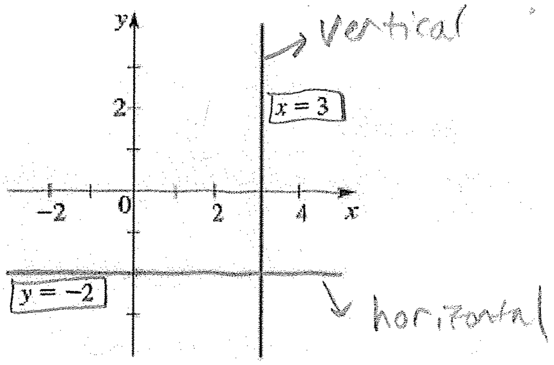
See ~~back of~~ next page for these (vertical and horizontal lines)

Vertical and Horizontal Lines

VERTICAL AND HORIZONTAL LINES

An equation of the vertical line through (a, b) is $\boxed{x = a}$

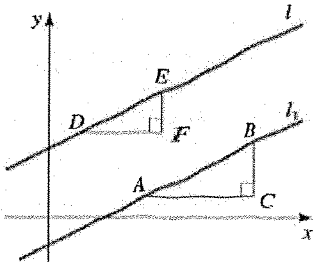
An equation of the horizontal line through (a, b) is $\boxed{y = b}$



Parallel Lines

PARALLEL LINES

Two nonvertical lines are parallel if and only if they have the same slope.



ex $y = \boxed{\frac{1}{2}}x - 6$

$(y-1) = \boxed{\frac{1}{2}}(x+3)$

same slope => parallel

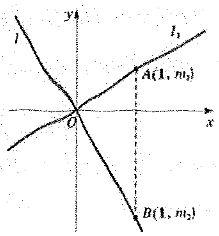
Perpendicular Lines

PERPENDICULAR LINES

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals:

$$\boxed{m_2 = -\frac{1}{m_1}}$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).



ex $y = 6x - 10$ $\qquad y = -\frac{1}{6}x + 30$

$M_1 = 6$ $\qquad M_2 = -\frac{1}{6}$

$M_1 \cdot M_2 = 6 \cdot -\frac{1}{6} = -1$

=> these are perpendicular

1) Determine the equation of the line that has a slope of 5 and that passes through (2 , 3). Use any type of equation you would like.

Point-slope

$y-3=5(x-2)$

Slope-Intercept

$3=5(2)+\boxed{b}$

$b=-7$

$\boxed{y=5x-7}$ →

General

$-5x+y+7=0$

2) Determine the equation of the line passing through (2 , -5) and (-4 , 3) in all 3 forms and then graph the line in the coordinate plane

① Slope $= \frac{3-(-5)}{-4-2} = \frac{8}{-6} = \boxed{-\frac{4}{3} = m}$

Point slope

$y-3=-\frac{4}{3}(x+4)$

$y+5=-\frac{4}{3}(x-2)$

Slope-Intercept

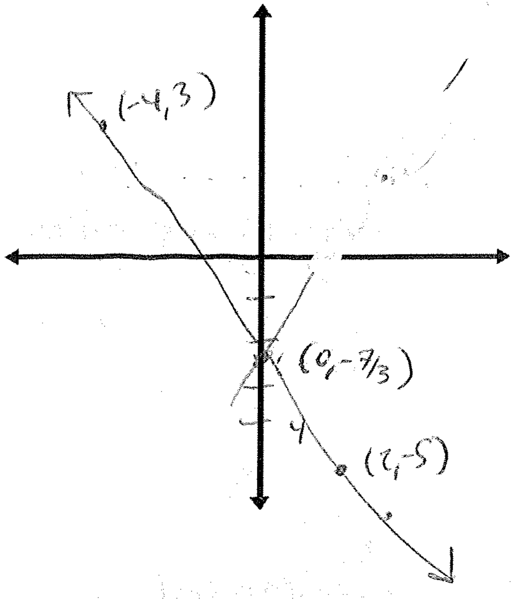
$3=-\frac{4}{3}(-4)+b$

$\boxed{b=-7/3}$

$y=-\frac{4}{3}x-7/3$

General

$\frac{4}{3}x+y+\frac{7}{3}=0$

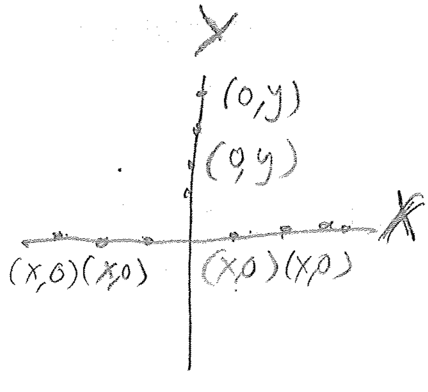


3) Determine the equation of the line in point-slope form that passes through (4 , 5) and has a slope of $-\frac{1}{2}$

~~$y-4=-\frac{1}{2}(x-5)$~~

$y-5=-1/2(x-4)$

4) Find the x-intercept and y-intercept of the$-3x-5y+30=0$



X-intercept

let $y=0$

$-3x+30=0$

$\boxed{x=10}$

y-intercept

let $x=0$

$-5y+30=0$

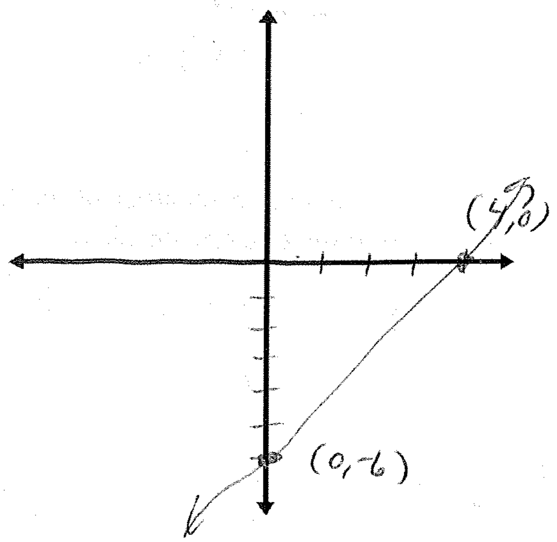
$\boxed{y=+6}$

5) Find the slope and y-intercept of $3x - 2y = 12$ and use them to graph the equation

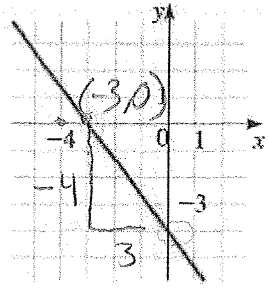
$y = \boxed{\frac{3}{2}} x - 6$

$M = \text{Slope} = \frac{3}{2}$

X-int	y-int
$y=0$	$x=0$
$3x=12$	$-2y=12$
$\boxed{x=4}$	$\boxed{y=-6}$



6) Determine the equation of the following line and write it in all 3 forms



$\text{Slope} = M = \frac{\text{rise}}{\text{run}} = \frac{-4}{3}$

Point slope

Use Point: (-3,0)

$y = -\frac{4}{3}(x+3)$

Slope-intercept

use point (-3,0)

$0 = -\frac{4}{3}(-3) + b$

$\boxed{b = -12/3}$

$y = \underset{M}{\underset{\uparrow}{-4/3}}\, x - \underset{b}{\underset{\uparrow}{12/3}}$

General

rewrite Slope-intercept

$y + 4/3\, x + 12/3 = 0$

or $3y + 4x + 12 = 0$

7) Determine the equation of the line that passes through $(\frac{1}{2}, -\frac{2}{3})$ and is perpendicular to $4x - 8y = 1$ in slope intercept form

rewrite $4x - 8y = 1$

as $y = \boxed{\frac{1}{2}} x - \frac{1}{8}$

$\Rightarrow M = \frac{1}{2}$

we want a line Perpendicular

So we want our $\boxed{M = -2}$

$\frac{1}{2} \cdot M = -1 \Rightarrow \boxed{M = -2}$

$y = -2x + b$

$-\frac{2}{3} = -2(\frac{1}{2}) + b$

$\boxed{b = 1/3}$

$\boxed{y = -2x + 1/3}$

(2013)

2.1- Functions

Our world is full of objects, concepts and forces. It is obvious that the state of many of these things have an effect on one another. That is, when one changes it affects the other.

Ex: Age and height- As a person's age changes so does their height. That is, your height depends on your age
↳ height is a function of age

Ex: Your grade in this class and the effort you put into the course- The more effort you put in the higher your average will be

Ex: gravity is a function of mass

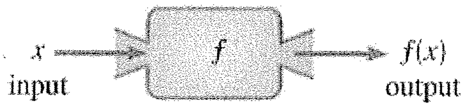
However, simply saying one thing depends on something else does not give any specific information about how the two are related. The best way to describe a relationship between two things is to use a function, which is something that associates one quantity (called the input) with another quantity (called the output).
↳ "x" ↳ "y" or "f(x)"

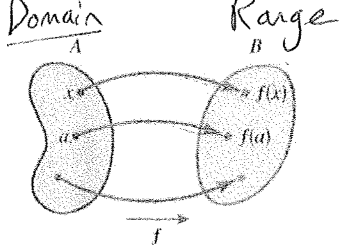
DEFINITION OF A FUNCTION

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

$f(x)=2x+1$

$x=2 \rightarrow 2x+1 \rightarrow 5=f(2)$





Usually sets A and B are sets of real numbers

We call set A (seen above) the domain- the set of all possible values of x that can be plugged into the function. Also known as the independent variable We this in Chapter 1

We call set B the range- the set of all possible values of f(x) that result from plugging in values from the domain. Also known as the dependent variable. In set-builder notation we can write the range as $\{f(x)|x\epsilon A\}$ → "All values we can get when we plug in the domain"

Basic Example: determine the domain and range of $f(x) = x^2 + 4$ and express them in interval notation

Domain: no restrictions
i.e., we can plug any value
into $f(x)=x^2+4$

Domain: $x \in \mathbb{R}$ → "x in real Numbers"

$\boxed{x \underset{\text{"in"}}{\in} (-\infty,\infty)}$

Range: Will there ever be a negative f(x) value?

No, since $x^{②}$ (+4) ⟹ always nonnegative
in fact, always ≥ 4

Range: $[4,\infty)$

Evaluating Functions

To evaluate a function at value simply substitute the value into the variable

Ex: If $\boxed{f(x) = 3x^2 - 4x + 1}$, evaluate the following

a) $f(3)$

$f(3) = 3(3)^2 - 4(3) + 1$

$= 3 \cdot 9 - 12 + 1$

$f(3) = 16$

b) $f(-a)$

$f(-a) = 3(-a)^2 - 4(-a) + 1$

$\boxed{f(-a) = 3a + 4a + 1}$

$\boxed{f(eqn^*) = 3(eqn^*)^2 - 4(eqn^*) + 1}$

c) $\dfrac{f(a+h) - f(a)}{h}$

$$\frac{f(a+h) - f(a)}{h} = \frac{\overset{f(a+h)}{[3(a+h)^2 - 4(a+h) + 1]} - \overset{f(a)}{[3(a)^2 - 4(a) + 1]}}{h}$$

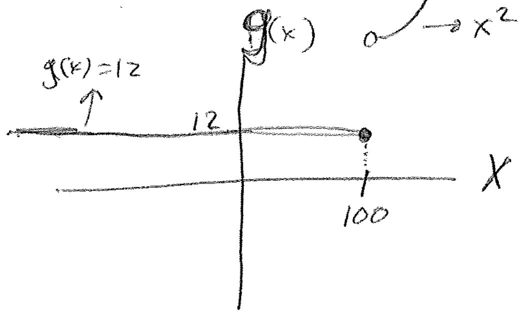
$$= \frac{[3(a^2 + 2ah + h^2) - 4a - 4h + 1] - [3a^2 - 4a + 1]}{h}$$

$$= \frac{[\cancel{3a^2} + 6ah + 3h^2 - \cancel{4a} - 4h + \cancel{1} - \cancel{3a^2} + \cancel{4a} - \cancel{1}]}{h}$$

$$= \frac{6ah + 3h^2 - 4h}{h} = \frac{h[6a + 3h - 4]}{h} = 6a + 3h - 4$$

Piecewise Functions- a function whose output values are broken into different pieces, each of which depends on the input value

Ex: $g(x) = \begin{cases} 12 & \text{If } -\infty \le x \le 100 \\ x^2 & \text{If } x > 100 \end{cases}$



g(27.9)= 12 → Since 27.9 ≤ 100

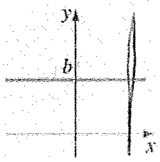
g(100,000)= $((100000))^2$ → Since 100,000 > 100

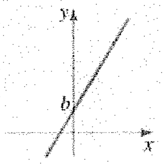
$\boxed{2013}$

2.2- Graphs of Functions

SOME FUNCTIONS AND THEIR GRAPHS

Linear functions
$f(x) = mx + b$

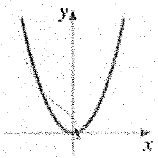

$f(x) = b$

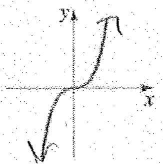

$f(x) = mx + b$

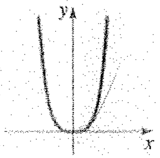
Power functions
$f(x) = x^n$

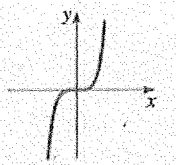
x^2

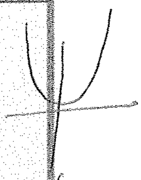
$x^3 + 2$

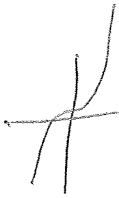

$f(x) = x^2$


$f(x) = x^3$

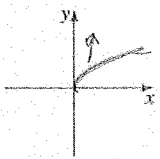

$f(x) = x^4$

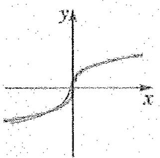

$f(x) = x^5$

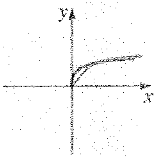

x^6

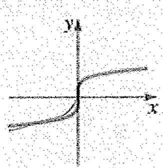

x^7

Root functions
$f(x) = \sqrt[n]{x}$

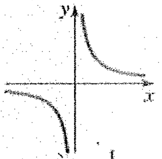

$f(x) = \sqrt{x}$

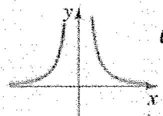

$f(x) = \sqrt[3]{x}$


$f(x) = \sqrt[4]{x}$


$f(x) = \sqrt[5]{x}$

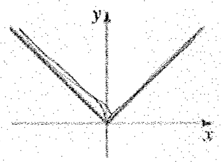
Reciprocal functions
$f(x) = \frac{1}{x^n}$


$f(x) = \frac{1}{x}$

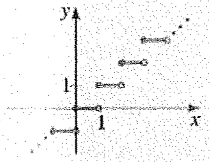

$f(x) = \frac{1}{x^2}$

$\frac{1}{(x+1)^2}$

Absolute value function
$f(x) = |x|$


$f(x) = |x|$

Greatest integer function
$f(x) = [\![x]\!]$


$f(x) = [\![x]\!]$

The shape of some common functions are shown above. When graphing a function we can either ⓐ use known information about its shape (2.5)

ⓑ use a table to plot points (2.2)

Graphing Functions by Plotting Points – Feel free to use a calculator to help fill out tables

Any function, especially those that are more complicated, can be *sketched* by making a table

To figure out what x-values to choose, need to know domain of the function

Use a table to Graph $f(x) = \sqrt{x+4}$

$x+4 \geq 0 \Rightarrow \boxed{x \geq -4}$

Domain: $\{x \mid x \geq -4\}$

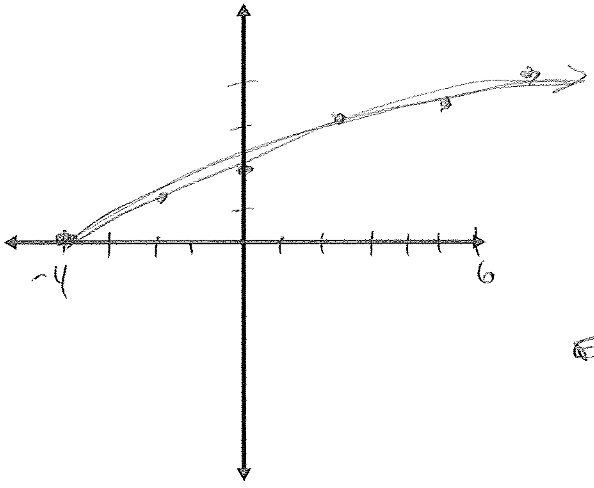
x	F(x)	(x,y)
-4	0	$(-4,0)$
-2	$\sqrt{2}$	$(-2,\sqrt{2})$
0	2	$(0,2)$
2	$\sqrt{6}$	$(2,\sqrt{6})$
4	$\sqrt{8}=2\sqrt{2}$	$(4,2\sqrt{2})$
6	$\sqrt{10}$	$(6,\sqrt{10})$

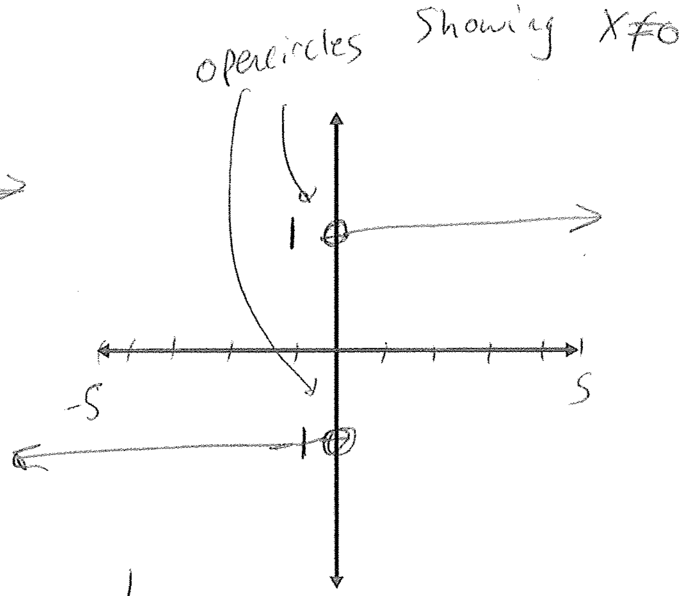
Use a table to Graph $f(x) = \frac{x}{|x|}$

Can't ÷ by 0

Domain: $\{x \mid x \neq 0\}$

x	F(x)	(x,y)
-5	$\frac{-5}{5}=-1$	$(-5,-1)$
-3	$-3/3=-1$	$(-3,-1)$
-1	-1	$(-1,-1)$
1	$1/1=1$	$(1,1)$
3	$3/3=1$	$(3,1)$
5	1	$(5,1)$





↳ piecewise function

$$f(x) = \begin{cases} -1 & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases}$$

Graphing Piecewise Functions

To graph a piecewise function you must consider what the domain of each piece is.

To draw the graph you can either use known information about the type of function or create a table for each piece of the function. (2.5) (2.2)

At the intersection points of the pieces, use an open circle or closed circle similar to drawing the graph of an interval. (depends on the inequality)

Graph $h(x) = \begin{cases} x^2 & \text{If } x < 4 \\ 4x - 9 & \text{If } x \geq 4 \end{cases}$

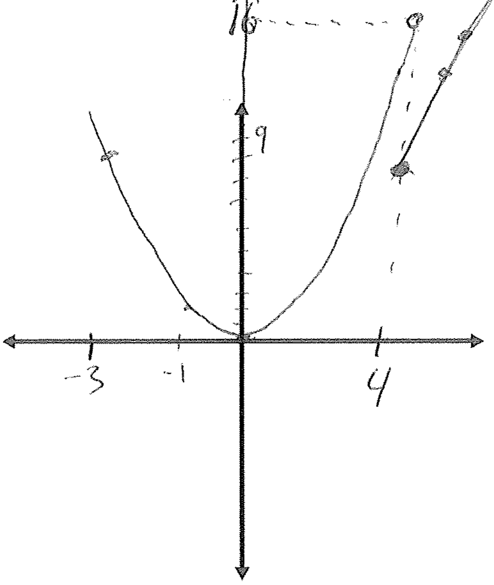
Graph $g(x) = \begin{cases} -1 & \text{If } x < -1 \\ x & \text{If } -1 \leq x \leq 1 \\ 1 & \text{If } x > 1 \end{cases}$

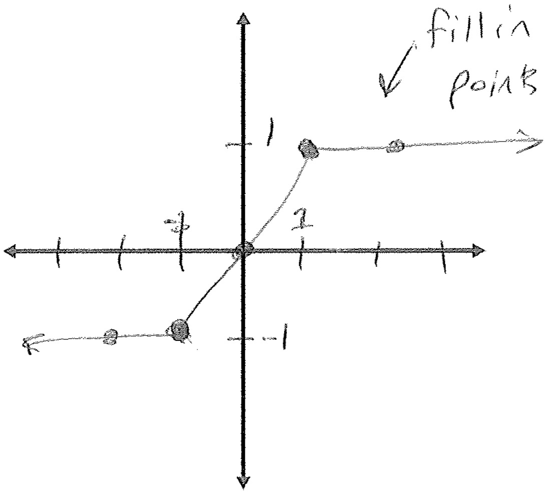
Domain: all real #s

Plot the intersection points in all pieces

	x	H(x)	(x,y)
x^2	-3	9	(-3,9)
	0	0	(0,0)
	4	16	(4,16)
4x-9	4	7	(4,7)
	5	11	(5,11)
	6	15	(6,15)

	x	G(x)	(x,y)
-1	-2	-1	(-2,-1)
	-1	-1	(-1,-1)
x	-1	-1	(-1,-1)
	0	0	(0,0)
	1	1	(1,1)
1	1	1	(1,1)
	2	1	(2,1)





fill in intersection point b/c graph pieces lines up with each other

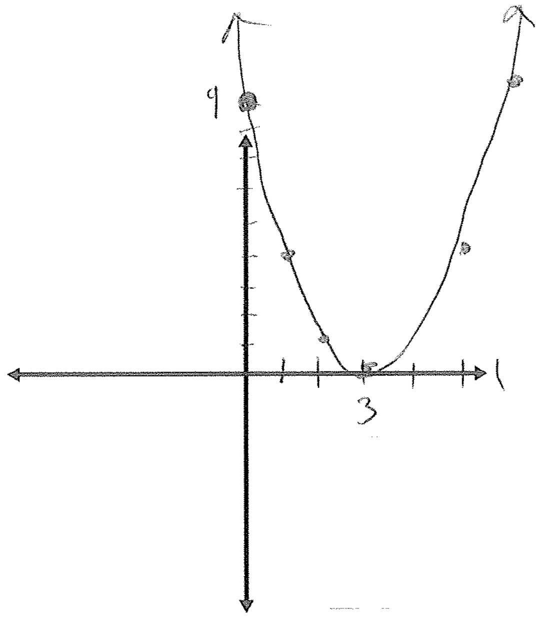
Graph $f(x) = (x-3)^2$

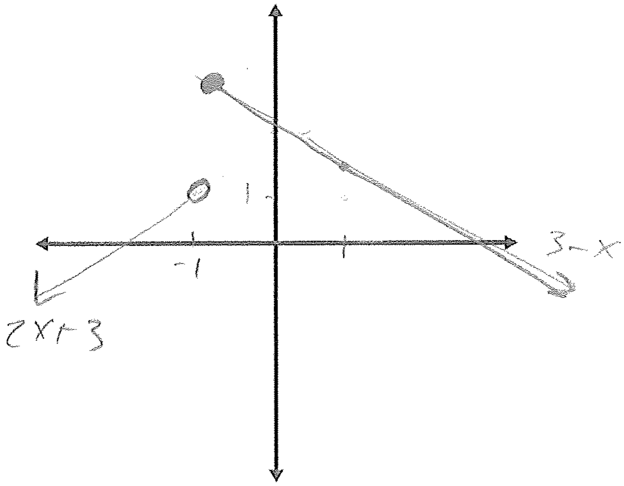
Graph $g(x) = \begin{cases} 2x+3 & \text{If } x < -1 \\ 3-x & \text{If } x \geq -1 \end{cases}$

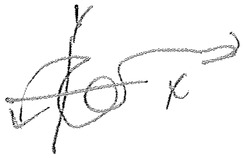
Domain: all x in real numbers

x	F(x)	(x,y)
0	9	(0,9)
1	4	(1,4)
2	1	(2,1)
3	0	(3,0)
5	4	(5,4)
6	9	(6,9)

	x	G(x)	(x,y)
2x+3	-3	-3	(-3,-3)
	-2	-1	(-2,-1)
	-1	1	(-1,1)
3-x	-1	4	(-1,4)
	0	3	(0,3)
	1	2	(1,2)
	2	1	(2,1)







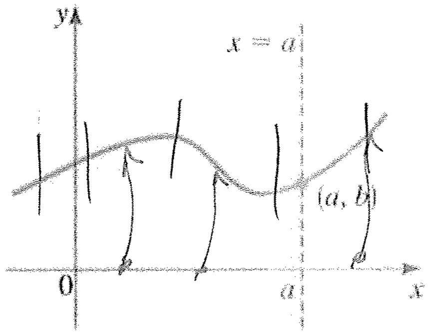
Vertical Line Test

Not all curves are functions. For every given value of x that we plug into a function it returns a single value, called f(x). If this fact is violated then the curve is not a function. The following test allows us to see if a curve in the plane is a function or not.

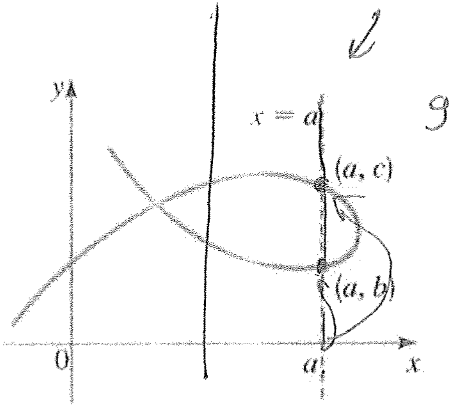
THE VERTICAL LINE TEST

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

Examples of functions: Each of the graphs seen thus far in this section



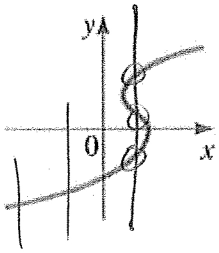
Graph of a function

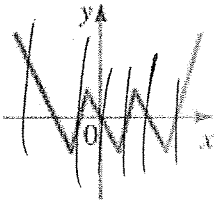


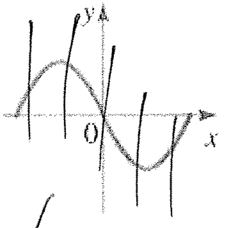
Not a graph of a function

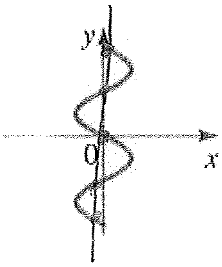
since vertical line intersects graph more than once not a function

Example: Use the vertical line test to determine which of the following curves are functions









Not a function

are functions by vertical line test

Not a function by vertical line test

2.3- Getting information From the Graph of a Function

Looking at the graph of a function is one of the best and easiest ways to get information about the function being depicted.

Domain and Range

Range- can be thought of as the "height" of a graph. That is, how far the graph extends in both the positive, as well as negative, y-direction

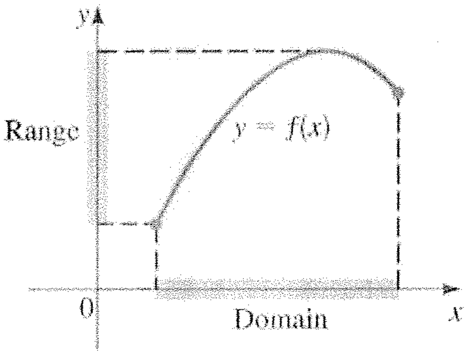
Domain- can be thought of as the "width" of a graph. That is, how far the graph extends in both the positive, as well as negative, x-direction

Reminder: we can write both the domain as well as the range is both set-builder notation ① as well as interval notation ②, keeping in mind the usual rules of "openness" and "closedness"

$g(x) = \bigcirc$

↑ x-value

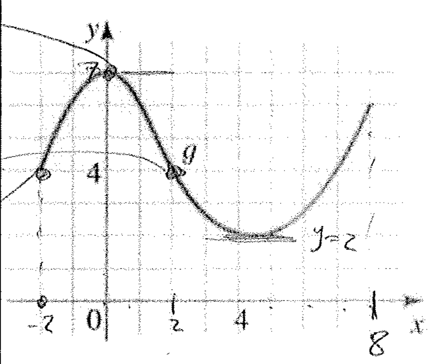
↑ corresponding y-value

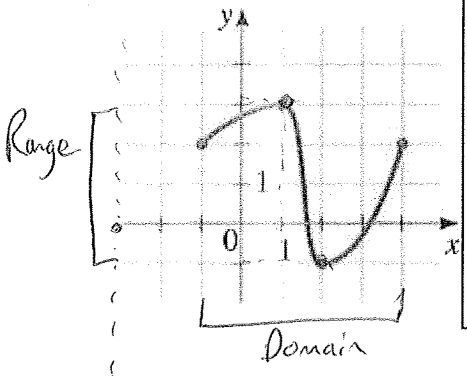


g(0)= 7

g(2)= 4

g(-2)= 4





f(1)= 3

f(2)= −1

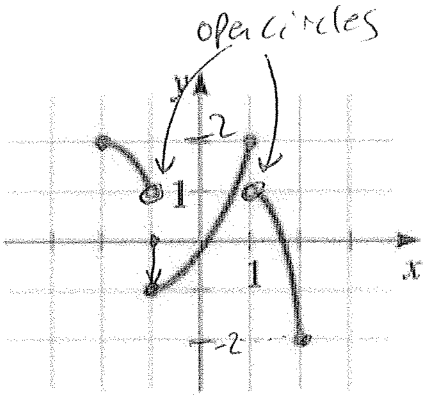
f(-3)= Does not exist (undefined)

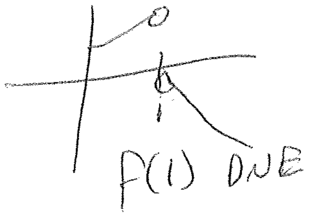
Range: $\{y \mid 2 \le y \le 7\}$ or $[2,7]$

Domain: $\{x \mid -2 \le x \le 8\}$ or $[-2,8]$

Range: $\{y \mid -1 \le y \le 3\}$ or $[-1,3]$

Domain: $\{x \mid -1 \le x \le 4\}$ or $[-1,4]$





Range: $\{y \mid -2 \leq y \leq 2\}$ or $y \in [-2, 2]$

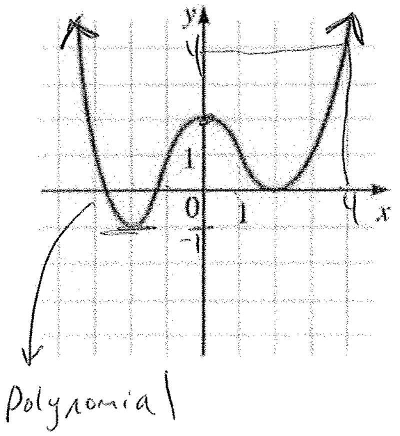
Domain: $\{x \mid -2 \leq x \leq 2\}$ or $x \in [-2, 2]$

f(-2)= 2

f(-1)= -1

f(1)= 2

(assume the graph continues in all directions)



$-1 \leq y \in \infty$

Range: $\{y \mid y \geq -1\}$ or $[-1, \infty)$

Domain: all real numbes $x \in (-\infty, \infty)$

all $x \in \mathbb{R}$

f(0)= 2

f(-2)= -1

f(4)= 4